An institutional evaluation of pension funds and life insurance companies
An institutional evaluation of pension funds and life insurance companies

Dirk Broeders, An Chen and Birgit Koos *

* Views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.
AN INSTITUTIONAL EVALUATION OF PENSION FUNDS AND LIFE INSURANCE COMPANIES

DIRK BROEDERS‡, AN CHEN*, AND BIRGIT KOOS§

Abstract. This paper compares two different types of annuity providers, i.e. defined benefit pension funds and life insurance companies. One of the key differences is that the residual risk in pension funds is collectively borne by the beneficiaries and the sponsor while in the case of life insurers, it is borne by the external shareholders. This paper employs a contingent claim approach to evaluate the risk return trade-off for annuitants. For that, we take into account the differences in contract specifications and in regulatory regimes. Mean-variance analysis is conducted to determine annuity choices of consumers with different preferences. Using realistic parameters we find that under linear and quadratic utility, life insurance companies always dominate pension funds, while under other utility specifications this is only true for low default probabilities. Furthermore, we find that power utility consumers are indifferent if the long term default probability of pension funds exceeds that of life insurers by 2 to 4%.

Keywords: Pension plans, barrier options, contingent claim approach, mean-variance analysis

JEL: G11, G23

Date: November 9, 2009.

‡ De Nederlandsche Bank (DNB), Supervisory Policy Division, Strategy Department, PO Box 98, 1000 AB Amsterdam, the Netherlands, Tel: 0031-20-5245794, Fax: 0031-20-5241885. E–Mail: dirk.broeders@dnb.nl. * Netspar and Department of Business Administration III, Faculty of Economics and Law, University of Bonn, Adenauerallee 24-42, 53113 Bonn, Germany. Tel: 0049-228-736103, Fax: 0049-228-735048, E–Mail: an.chen@uni-bonn.de. § Department of Business Administration III, Faculty of Economics and Law, University of Bonn, Adenauerallee 24-42, 53113 Bonn, Germany. Tel: 0049-228-739229, Fax: 0049-228-735048, E–Mail: birgit.koos@uni-bonn.de. The authors thank Paul Cavelaars, Theo Nijman, Frans de Roon, participants of the seminar at DNB (September 1, 2009) and the 16th Annual Meeting of the German Finance Association (October 9 and 10, 2009) for helpful comments. The views expressed in this paper are personal and do not necessarily reflect those of DNB.
Table 1. Total size of funds under management by region; Source: OECD totals in USD billion for 2006.

<table>
<thead>
<tr>
<th>Region</th>
<th>Pension funds</th>
<th>Life insurers</th>
</tr>
</thead>
<tbody>
<tr>
<td>North America</td>
<td>10,400</td>
<td>3,972</td>
</tr>
<tr>
<td>Europe</td>
<td>2,169</td>
<td>4,301</td>
</tr>
<tr>
<td>U.K.</td>
<td>1,831</td>
<td>2,562</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>14,400</strong></td>
<td><strong>10,835</strong></td>
</tr>
</tbody>
</table>

1. Introduction

Defined benefit pension funds and life insurance companies are both key annuity providers. Besides governments, they are important institutions in the world for arranging old age income provisions efficiently. Table 1 shows the importance of pension funds and life insurers by the size of assets under management for North America, Europe and the U.K. In North America pension funds dominate life insurance companies, while in Europe and in the U.K. life insurers appear to be more important.

Although they offer similar products, there are also distinct institutional differences between pension funds and life insurance companies, as illustrated in Table 2. Pension funds usually have the form of non-profit organizations or trusts. A defined benefit pension provides a life-long income after retirement based upon the years of service, salary and a certain accrual rate, see Bodie (1990). The pension benefit might be (conditionally) indexed to inflation or wages, see Bikker and Vlaar (2007). Over their careers, part of employees’ compensation is transferred to a pension fund that manages the assets and the liabilities. The available surplus (the difference between assets and liabilities) can be regarded as the pension fund’s equity. It acts as a risk buffer. The residual risk is partly borne by the beneficiaries themselves. The collective beneficiaries act as bondholders and shareholders. In a continuum of overlapping generations this also includes future participants. New entrants to a pension fund might be confronted with losses (or gains) that accrued in the previous period. Often also the company behind the pension fund is either explicitly or implicitly involved in risk sharing. For instance, in case of underfunding (or in case of overfunding) contributions to the pension fund can be increased (or decreased).

1Residual or net risk is the gross risk exposure minus risk mitigation procedures.
In case of pension guarantee funds, like the Pension Benefit Guarantee Corporation in the United States or the Pension Protection Fund in the U.K., the residual risk is also covered by these institutions. Specifically, in the case the corporation defaults, pension rights are protected by these guarantee schemes. The pension fund’s board typically consists of representatives of both employers and employees. The board decides on asset allocation, contribution rate and indexation policy. Beneficiaries can have some influence on this through the election of board members.

For-profit life insurance companies, usually in the form of incorporations (Inc), also provide annuities. A life annuity is a financial contract in the form of an insurance product

<table>
<thead>
<tr>
<th>Pension fund</th>
<th>Insurance company</th>
</tr>
</thead>
<tbody>
<tr>
<td>Institutional structure</td>
<td>Trust</td>
</tr>
<tr>
<td>Who bears residual risk?</td>
<td>(Future) beneficiaries are residual claim holders, risk sharing with sponsor</td>
</tr>
<tr>
<td>Contract specifications</td>
<td>Defined benefit pension</td>
</tr>
<tr>
<td>- Guaranteed benefit</td>
<td>Contingent indexation related to inflation or wage growth</td>
</tr>
<tr>
<td>- Indexation policy</td>
<td>Typically large mismatch</td>
</tr>
<tr>
<td>Investment policy</td>
<td>Typically low, e.g. 97.5%</td>
</tr>
<tr>
<td>Regulation parameters</td>
<td>Typically high, e.g. 3 years</td>
</tr>
<tr>
<td>- Confidence level</td>
<td>Contribution rate</td>
</tr>
<tr>
<td>- Recovery period</td>
<td>Asset allocation</td>
</tr>
<tr>
<td>Policy variables</td>
<td>Indexation policy</td>
</tr>
<tr>
<td></td>
<td>Initial surplus</td>
</tr>
</tbody>
</table>

Table 2. Typical differences between pension funds and life insurers

---

2An insurance company can also take the form of a mutual. This is fairly comparable to a pension fund.
according to which a life insurance company makes a series of payments in the future to
the buyer in exchange for an immediate lump sum payment. The payment stream con-
tinues until the date of death of the annuitant. The annuity might be increased with
annual bonuses if the underlying investments deliver sufficient returns, a feature called
and 2007b). The policy holders participate to a certain extent in the wealth of the life
insurer. Here, beneficiaries act as bond holders while shareholders provide equity and ac-
cept the residual risk. E.g. if the return on assets of the insurance company is negative or
if there is an underwriting loss, this will be absorbed by the shareholders. On the other
hand, if performance is above average the shareholders are entitled to participating in the
surpluses. However, shareholders have limited liability. In case of default, the shareholders
will not lose more than their initial investment. As shareholders are residual claimants as
well as owners, they decide on the investment policy, insurance premiums, the with-profit
policy and capital structure. The policy holders have no say in this.

In addition, there are distinct differences in investment policy between pension funds
and life insurance companies. On average, pension funds run a larger mismatch risk com-
pared to life insurance companies.\textsuperscript{3} This is shown in Table 3. Pension funds in the United
States, Europe and the U.K. invest more heavily in equities. Insurance companies are more
focused on asset and liability matching and prefer fixed-income assets. This difference in
investment strategy is probably best explained by the differences in risk preferences of the
financial institutions, which also appear in diverging regulatory procedures.

In order to make annuity payments, pension funds and insurance companies are gen-
ernally subject to the, so called, full funding requirement. This means that at all times
the value of assets should at least be equal to the value of liabilities including a small
margin, i.e. the funding ratio should always be in excess of 105\%.\textsuperscript{4} Some countries, like
the Netherlands, explicitly prescribe that pension funds should hold additional regulatory

\textsuperscript{3}Mismatch risk typically occurs if the risk profile of the assets is very different from the risk profile of
the liabilities.

\textsuperscript{4}For the purposes of calculating the minimum amount of the additional assets, the European Pension
minimum amount is 4\% of the technical provisions plus 0.3\% of the capital at risk.
funds, which are necessary to absorb short-term deviations in the funding ratio, which is defined as the ratio of assets to liabilities. The required surplus is usually a function of the level of mismatch risk between assets and liabilities. A high mismatch risk requires a large surplus, and in case of asset-liability matching the surplus can be kept to a minimum. Life insurance companies are also required to maintain sufficient capital. However, in current European legislation (“Solvency I”) the required solvency margin currently does not depend on asset liability mismatches. This will change in the future “Solvency II” framework for insurers.

Pension regulation nonetheless is less strict. The confidence level of 97.5\%, e.g. for Dutch pension funds is significantly lower than the 99.9\% confidence level in “Solvency II”. In addition, some countries allow substantial recovery periods for pension funds to restore sufficient funding, see Broeders and Chen (2008). Insurance supervision on the other hand is stricter. If the solvency ratio (available over required solvency level) is inadequate the supervisory authorities will react promptly and the life insurance company will be liquidated if there is no resurrection in the short run. This way, consumers are relatively certain that they do not lose significant value at liquidation. The differences

<table>
<thead>
<tr>
<th>Region</th>
<th>Pension funds</th>
<th>Life insurers</th>
</tr>
</thead>
<tbody>
<tr>
<td>North America</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equities</td>
<td>62%</td>
<td>4%</td>
</tr>
<tr>
<td>Bonds</td>
<td>33%</td>
<td>78%</td>
</tr>
<tr>
<td>Other</td>
<td>5%</td>
<td>18%</td>
</tr>
<tr>
<td>Europe</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equities</td>
<td>40%</td>
<td>26%</td>
</tr>
<tr>
<td>Bonds</td>
<td>50%</td>
<td>59%</td>
</tr>
<tr>
<td>Other</td>
<td>10%</td>
<td>16%</td>
</tr>
<tr>
<td>U.K.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equities</td>
<td>66%</td>
<td>43%</td>
</tr>
<tr>
<td>Bonds</td>
<td>25%</td>
<td>39%</td>
</tr>
<tr>
<td>Other</td>
<td>9%</td>
<td>18%</td>
</tr>
</tbody>
</table>

Table 3. Asset allocation by region; Source: IMF, OECD, 2006
in regulation may be explained by the additional policy instruments that pension funds possess, such as their ability to renegotiate and raise future contributions\textsuperscript{5}. As an ultimate measure pension funds can also reduce accrued benefits to restore its funding level\textsuperscript{6}. As a rule of thumb, this greater flexibility should therefore reflect more or less the difference in confidence levels and recovery periods.

The present paper aims to make a comparative analysis of the two institutions at hand. \textit{First}, we specify the financial contract provided by the two institutions. We employ a contingent claim approach to value these contracts, taking account of default risk explicitly. We can already observe the difference between these two institutions by looking at fair combinations of contract parameters, particularly the fair participation rate with which contract holders are allowed to participate in the surpluses of the institutions. This part of the analysis is based on a risk-neutral valuation of the contracts at hand. \textit{Second}, we investigate in the physical world under which assumptions an annuitant prefers a pension fund over a life insurer. In order to answer this question, we look at the expected value and the variance of the terminal contract payoff for various consumers’ preferences.\textsuperscript{7}

This paper adds to the existing literature in two ways. First, we make a rigorous comparison between defined benefit pension funds and life insurance companies using a contingent claims approach. This allows for the construction of fair contracts. Second, we employ a utility-based mean-variance approach to test under which realistic parameters either the pension fund or the life insurance company is preferred.

There are already several papers that discuss the similarities and differences between pension funds and life insurance companies. Blake (1999), for example, draws parallels associated to the long-term nature of liabilities and investment objective. The not-for-profit pension funds do not attract funding in a competitive market, but seek to meet pension obligations at minimum cost to the scheme’s sponsor. Typically, life insurance companies

\textsuperscript{5}This feature is enhanced in case the pension scheme is (quasi) mandatory.  
\textsuperscript{6}An insurance company cannot reduce benefits as this would trigger liquidation.  
\textsuperscript{7}Since Chen and Suchanecki (2007) and Broeders and Chen (2008) provide a detailed analysis on the recovery period, we ignore the effects of the recovery period in this paper.
need to raise funding in a competitive market and as such have additional costs in the form of marketing expenditures. They are in the so-called “spread business” as they try to earn a spread on the return on assets and funding costs and on the underwriting of insurance risks.

Davis (2002) describes distinct differences in the risks that both institutions face, reflected in their investment strategies. Pension liabilities are typically more uncertain than those of life insurance companies. Defined benefit pension liabilities are related to wage growth during the accumulation phase and often linked to inflation after retirement. These differences reflect in a more profound investment strategy. Pension funds favor real investment opportunities that keep track with the development of liabilities. Often, stocks and real estate are considered the best investment choice for that.\(^8\) Life insurance companies often prefer bonds as the early surrender option can reduce the duration of liabilities significantly.\(^9\) Next to diverging investment policies, Davis (2002) also argues there are enough reasons for different regulatory regimes. As pension funds have a greater need for real investment returns they require more flexibility on the asset side. While life insurance companies, operating in a competitive market should be supervised more strictly as they are more likely to make “errors” in premia due to competitive pressures.

The remainder of the paper is organized as follows. Section 2 introduces the contracts provided both by pension funds and life insurance companies. Section 3 investigates the valuation of the contracts, which serves the fair contract analysis with some illustrations of numerical results. Section 4 conducts a mean-variance analysis, using different utility specifications and realistic parameters, to find out which institutions are preferred by potential annuity buyers. Section 5 concludes the paper.

---

\(^8\)Several papers challenge this conventional wisdom. Exley et al. (1997), Bader and Gold (2003) and Gold and Hudson (2003) all apply the no-arbitrage principle and the law of one price to show that the higher expected return on stocks reflects their greater risk in such a way that the risk-adjusted expected returns of stocks is equal to the return on risk-free bonds. Bodie (1995) demonstrates that insurance against an overall return below the risk-free interest rate may be acquired by a “forward-strike” put option. The crucial insight is that the value of the put option is shown to increase with time to maturity and volatility.

\(^9\)Early surrender is the right the policy holder has to cancel the contract prematurely.
2. Contract specification

This section introduces the general contract payoffs. For this, we assume that the pension fund operates for a single homogeneous class of beneficiaries who have to work another $T$ years. The life insurer has a single homogeneous class of policyholders who have bought a deferred annuity becoming due in $T$ years. This simplifying assumption is justified by the observation in practice that these financial institutions often base their financial strategy on the average policyholder’s characteristics. In addition, we assume that the annuity is paid out as a single cash flow at maturity. The minimum amount for this cash flow will be denoted by $L$ and is the same in both cases. This cash flow is equal to the present value of the annuity payments over the expected remaining life of the annuitant. All idiosyncratic mortality risk is diversified by the homogeneous class of policyholders and the aggregate mortality risk is fully priced in the annuity.\(^{10}\) We distinguish between the contract’s payoff at maturity and the payoff when premature default occurs. Such premature default is triggered if the regulatory minimum boundary is crossed. Default immediately means liquidation of the institution on hand.

2.1. Payoff in case of no premature default. First we define the payoff at maturity. That is to say, in case no premature default occurs.

2.1.1. Defined benefit pension fund. At time 0, the pension fund issues a defined benefit pension plan to a representative group of beneficiaries who provide an upfront contribution $P_0$. The pension fund also receives an initial amount of money from the sponsor $S_0$ at time 0. The sponsor’s contribution shows he is involved in a risk sharing arrangement with the beneficiaries. Consequently, the initial asset value of the pension fund is given by the sum of the contributions from both the beneficiaries and the sponsor, i.e. $A_0 = P_0 + S_0$. From now on, we shall denote the contribution a proportion of the initial asset value, or $P_0 = \alpha A_0$ with $\alpha \in [0, 1]$. The pension fund invests the total proceeds in a diversified portfolio of risky and non-risky assets.\(^{11}\)

\(^{10}\)See e.g. Brown and Orszag (2006) for a further description of idiosyncratic and aggregate mortality risk.

\(^{11}\)We hereby implicitly assume that there is no liquid market for inflation linked and longevity products that would allow the pension fund to replicate its liabilities in the capital market.
At retirement $T$, the beneficiaries receive a lump sum nominal pension of $L$. In addition, the pension plan has the objective of increasing pension rights by $i\%$ per annum, where $i\%$ might be related to, say, the average expected CPI or wage growth over the life of the contract. Since the determination of this parameter has to take into consideration many factors, in reality this procedure is fairly complicated. Here, for simplicity’s sake, we assume $i$ is deterministic and a fully indexed pension is then equal to $\bar{L} = Le^{iT}$. However, it should be noted that the actual outcome of the pension plan is contingent on the funding ratio at maturity $T$. Therefore, the contract is conditionally indexed. At maturity $T$, given that the assets are sufficiently high ($A_T > \bar{L}$), the beneficiaries not only receive a fully indexed pension of $\bar{L}$, but are also allowed to participate in the pension fund’s surplus ($A_T - \bar{L}$) with a participation rate $\delta$, where $\delta \in [0, 1]$. This parameter can be identified as the surplus distribution parameter or participation rate. For instance, $\delta = 0.25$ means that the pension beneficiary receives $1/4$ of the surplus and the sponsor $3/4$. When the assets of the pension fund do not perform well enough to cover a fully indexed pension, we distinguish between two scenarios: $A_T < L$ and $L \leq A_T < \bar{L}$. In the latter case, the full assets value $A_T$ is assigned to the beneficiaries, whereas in the former case the guaranteed amount $L$ is paid out to them. Since a pension fund does not have external shareholders, instead the corporate pension plan sponsor\textsuperscript{12} covers the deficit at maturity in case the assets are insufficient to pay out the guaranteed benefit $L$. In short, at maturity date $T$ – assuming no early termination – the payoff to the beneficiary is:

$$
\psi_B(A_T) = \begin{cases} 
L, & \text{if } A_T < L \\
A_T, & \text{if } L \leq A_T < \bar{L} \\
\bar{L} + \delta(A_T - \bar{L}), & \text{if } A_T > \bar{L}.
\end{cases}
$$

More compactly, we can rephrase this payoff as

$$
\psi_B(A_T) = L + [A_T - L]^+ + (1 - \delta)(A_T - \bar{L})^+,
$$

where we have used $[x]^+ := \max\{x, 0\}$. This payoff consists of three parts: a promised amount $L$, a long call option on the assets with strike equal to the promised payment $L$, and a short call option with strike equal to $\bar{L}$ (multiplied by $1 - \delta$). The latter represents the money returned to the pension plan sponsor. This is a reward the beneficiaries are

\textsuperscript{12}Besides the corporate plan sponsor, the guarantee could also be provided by a pension guarantee fund or the government.
willing to pay to the sponsor for covering the shortfall risk.

Including this compensation, the total payoff to the pension plan sponsor at maturity
\( \psi_S(A_T) \), is given by:

\[
\psi_S(A_T) = \begin{cases} 
A_T - L, & \text{if } A_T < L \\
0, & \text{if } L \leq A_T \leq \bar{L} \\
(1 - \delta)(A_T - \bar{L}), & \text{if } A_T > \bar{L}
\end{cases}
\]

or more compactly,

\[
\psi_S(A_T) = (1 - \delta)[A_T - \bar{L}]^+ - [L - A_T]^+.
\]

This payoff can be decomposed into two terms: a long call option which corresponds to the “bonus” received by the sponsor in prosperous times, and a short put option reflecting the deficit which he covers in case of underperformance of the assets. The payoffs to the beneficiary and the sponsor as a function of the asset value at maturity are illustrated in Figure 1.

2.1.2. Life insurance company. A for-profit insurance company, usually in the form of an incorporation (Inc.), can also provide annuities. The beneficiaries act as the bond holders in this situation while the shareholders provide equity. A (simplified) initial capital structure of a for-profit insurance company is described in Table 4. That is, for simplicity, we suppose that the homogeneous group of policyholders, whose premium payment at the beginning of the contract constitutes the liability of the insurance company, denoted by \( P_0 = \alpha A_0 \),
<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>Equity ($E_0 = A_0 - P_0 = (1 - \alpha)A_0$)</td>
</tr>
<tr>
<td></td>
<td>Liabilities ($P_0 = \alpha A_0$)</td>
</tr>
</tbody>
</table>

Table 4. Initial capital structure of a life insurance company

$\alpha \in [0, 1]$ (the same as in the pension fund framework), and the representative shareholder whose equity is accordingly denoted by $E_0 = (1 - \alpha)A_0$ together form a life insurance company. In practice a commercial life-insurer will have to make marketing expenditures. We neglect those in our analysis.

At the maturity date $T$, the outstanding liability that the insurance company should redeem to the beneficiary who survives time $T$ is given by

$$
\psi_L(A_T) = \begin{cases} 
A_T & \text{if } A_T < L \\
L & \text{if } L \leq A_T \leq \frac{L}{\alpha} \\
L + \delta_L (\alpha A_T - L) & \text{if } A_T > \frac{L}{\alpha}
\end{cases}
$$

or more compactly

$$
\psi_L(A_T) = L + \delta_L \alpha \left[ A_T - \frac{L}{\alpha} \right]^+ - \left[ L - A_T \right]^+ \tag{2}
$$

with $A_T$ here denoting the insurance company’s asset value at time $T$. Figure 2 plots the terminal payoff for the policyholder and the shareholder. As illustrated in Figures 1 and 2, the payoffs of the beneficiary in a defined benefit pension fund and in a life insurance company framework show quite different patterns. More specifically, when the final asset’s value is not sufficiently high ($A_T < L$), in the insurance contract, the contract holder will obtain $A_T$ due to the limited liability of the shareholder, whereas in a pension plan, a floor (here $L$, provided by the sponsor) is ensured to the beneficiary. The shareholders accept the residual risk but will not lose more than their initial investment. Therefore, when the firm’s assets are underperforming, only the existing assets $A_T$ will be assigned to the surviving beneficiaries. This implies that the for-profit insurance company in fact does not provide the beneficiary with a guaranteed amount $L$ under all circumstances. When the assets perform moderately ($L \leq A_T \leq \frac{L}{\alpha}$), a with-profit life insurance provides its...
contract holder with the guaranteed amount $L$, whereas in a pension plan, the entire asset value is assigned to the beneficiary. Finally, if the assets perform well ($A_T > L$), both pension fund and life insurance company allow their contract holders to participate in their surplus. The differences are twofold. One lies in the triggering condition for surplus sharing. In a pension fund framework, the surplus is partly awarded to the beneficiary when the asset value exceeds the fully indexed pension $\bar{L}$, whereas in the case of a for-profit insurance company, the trigger is at $\frac{L}{\alpha}$. The second difference concerns the participation rate, which corresponds to $\delta$ in a pension fund and $\delta_L \alpha$ in a for-profit insurance company framework. This means that the bonus payment depends on the (initial) contribution rate of the policyholder in the case of a for-profit life insurance company.

When the performance of the firm’s assets is above average, the shareholder will benefit. The payoff of the shareholder is given by

$$\psi_E(A_T) = [A_T - L]^+ - \delta_L \alpha \left[A_T - \frac{L}{\alpha}\right]^+. \quad (3)$$

The payoff of the shareholder is also given in Figure 2.

2.2. Premature default formulation. We now look at a situation in which premature liquidation is enforced by the regulator. We suppose that the regulator monitors the firm’s asset value $A_t$ continuously because a company has to be solvent at any time.\textsuperscript{13} The

\textsuperscript{13}In particular, in the case of continuous monitoring by the regulator, when policies also include surrender options, the company should be able to give back the promised amount at any time.
regulator gives notice of default and and orders liquidation\textsuperscript{14} when the insurer’s firm assets $A_t$ become too low, mathematically when they hit some deterministic time-dependent barrier $B_t$:

$$B_t = \eta \underline{L} e^{-r(T-t)}$$

(4)

for $t \in [0,T]$, where $r$ is the risk-free interest rate. This parameter $\eta$ may be regarded as a regulation parameter controlling the strictness of the regulation rule. As noted in Footnote 3, an $\eta$ value of 1.05 is the most appropriate level in practice, at least for insurers. For pension funds it is officially also the regulatory minimum although in practice it does not immediately invoke liquidation. Liquidation time $\tau$ is given by

$$\tau = \inf \{ t \in [0,T] \mid A_t \leq B_t \}.$$  

(5)

If $\tau \leq T$, the result is premature liquidation. The liquidation time is constructed as the first time that the firm’s level of assets hits the barrier from above. We do not take into account that a pension fund can increase contributions in case of a funding shortfall. Upon premature liquidation, a rebate payment

$$\Theta_B(\tau) = \Theta_L(\tau) = \min \{ \eta, 1 \} \underline{L} e^{-r(T-\tau)}$$

is provided to the beneficiaries of both the pension fund and the life insurer. The pension fund’s sponsor or the life insurer’s shareholders receive the residual when there is any (for $\eta \leq 1$, they receive nothing), i.e.

$$\Theta_S(\tau) = \Theta_E(\tau) = \max \{ \eta - 1, 0 \} \underline{L} e^{-r(T-\tau)}.$$  

(7)

Hereby we have ignored liquidation costs and other types of costs and distortions.

\textbf{3. Fair Contract Analysis}

The first question which comes to mind given the setting above is which combination of parameters offers fair contracts. The pension fund and the life insurance company can basically influence two parameters: the surplus participation rate ($\delta, \delta_L$) and the risk profile of the assets ($\sigma$). Other parameters, like $\underline{L}, \bar{L},$ and $\alpha$, are assumed exogenous for the financial institutions.

\textsuperscript{14}For instance, in Chapter 7 Bankruptcy Procedure of the U.S. Bankruptcy Code, default leads to immediate liquidation.
3.1. **Theoretical analysis.** The principle of a fair contract can be addressed to answer this question. Similar to Grosen and Jørgensen (2002) and Broeders and Chen (2008), a fair contract is determined when the market value of each stakeholder coincides with its initial contribution to the pension fund or life insurance company. The market value corresponds to the expected discounted payment under the unique equivalent martingale measure $P^*$, i.e.

$$V_k(A_0) = \frac{E^* \left[ e^{-rT} \psi_k(A_T) 1_{\{\tau > T\}} \right]}{\text{market value of the payoff at maturity}} + \frac{E^* \left[ e^{-r\tau} \Theta_k(\tau) 1_{\{\tau \leq T\}} \right]}{\text{market value of the rebate payment}} , \quad k = B, S, L, E,$$

where $B, S, L, E$ stand for beneficiary, sponsor, policyholder, and shareholder respectively. $E^*$ is used to denote the expectation taken under this measure. Here we accumulate the rebate payment until maturity date $T$ for consistency reasons. For the valuation, the assets are assumed to evolve according to

$$dA_t = A_t (r dt + \sigma dW^*_t)$$

where $\sigma$ is the instantaneous rate of return of the asset under the risk-neutral probability measure $P^*$ while $W^*_t$ is a standard Brownian motion under this measure.

Value $V_k(A_0)$ ($k = B, L, S, E$) consists of the time 0 value of several components. Determining the market values of each stakeholder’s claim boils down to deriving several values: the value of the fixed payment $L$ given that there is no default during $[0,T]$ ($\tau > T$), a down-and-out call option, a down-and-out put option and the expected discounted rebate payment if there is a default ($\tau \leq T$). More specifically, we have

$$V_B(A_0) = FP(L) + DOC(L) - (1 - \delta)DOC(\bar{L}) + RP_B$$
$$V_S(A_0) = (1 - \delta)DOC(\bar{L}) - DOP(L) + RP_S$$
$$V_L(A_0) = FP(L) + \delta L \alpha DOC(\bar{L}/\alpha) - DOP(L) + RP_L$$
$$V_E(A_0) = DOC(\bar{L}) - \delta L \alpha DOC(\bar{L}/\alpha) + RP_E.$$

Each component is explained in detail in Appendix 5.1.

3.2. **Numerical illustration.** The valuation formula obtained in the last subsection (specified in Appendix 5.1) is implemented here in order to conduct a fair contract analysis.
Figure 3. Fair combinations of $\sigma$ and $\delta$ (or $\delta_L$) for different values of $\eta$ (with “Pe” and “Li” standing for pension fund and life insurance, respectively).

The parameters are set as follows:

\[
\begin{align*}
    r &= 5\%, \quad L = 120, \quad i = 2.5\%, \quad A_0 = 100, \quad T = 15, \quad \alpha = 0.9 \quad \text{and} \quad \eta = 1.05.
\end{align*}
\]

The parameters are realistic and have been chosen to reflect current market conditions. An $i$ of 2.5% is an appropriate index for the pension fund case as pension benefits are often indexed to inflation or wage growth. For $\alpha = 0.9$ the initial investment of the beneficiaries (or the liability holders) is 90. $L = 120$ implies that a guaranteed interest rate of 1.9% is provided to the beneficiary (or the liability holder). As interpreted in footnote 3, $\eta$ shall be close to 1.05, which indicates an initial barrier level of 59.52.

Figure 3 illustrates fair combinations between the asset return volatility $\sigma$ and participation rate $\delta$ (pension fund case) or $\alpha \delta_L$ (life insurance company case) for different $\eta$ levels. Since we assume that both the beneficiary and the policyholder obtain a rebate payment corresponding to the barrier level at the default time, they own the same value when we assign the same value to $\eta$ (or barrier level). In other words, the market value of the benefits of the beneficiary and that of the policyholder only differ in the contract payoff when there is no premature default. This allows us to focus our comparison on the effect of investment policy and contract specification. Figure 3 allows for several observations.

*First*, it is observed that, overall, the fair participation rate for pension funds ($\delta$) is lower than that for life insurers ($\alpha \delta_L$). It is just due to the fact that we assumed that the sponsor ensures the beneficiary a guaranteed amount $L$ at maturity, whereas the insurer only has limited liability, which implies that only the remaining asset value $A_T$ is provided to the policyholder when underfunding ($A_T < \bar{L}$) occurs at maturity $T$. As a compensation, $\alpha \delta_L$ should be larger than $\delta$ in order to make the contract fair. In other words, fairness
implies that the insurance company offers a higher participation rate to its clients than the pension fund. Second, we take a look at the effect of the risk level on the contract value, and consequently on the fair participation rate. As volatility goes up, the value of the down-and-out call increases, while the value of the down-and-out put first increases with volatility and then decreases (hump-shaped). The value of the fixed payment goes down and the rebate term behaves similarly towards the down-and-out put, i.e., it first goes up and after a certain level of volatility is reached it goes down. This implies that the total effect of $\sigma$ on the contract value is indeed ambiguous. Here, in the framework of a pension fund, for the given parameters, the contract value decreases in $\sigma$. Hence there is a positive relation between $\delta$ and $\sigma$. In contrast, the contract value of the policyholder (life insurance case) goes up in $\sigma$, which leads to a negative relation between $\delta_L$ and $\sigma$. Third, the effect of $\eta$ (the regulation parameter) can be observed from the figures, too. Different $\eta$-values lead to different values of the barrier ($B_t = \eta L e^{-r(T-t)}$). As $\eta$ (the barrier) is set higher, the values of the down-and-out call and put decrease and so does the value of the fixed payment. In contrast, the expected value of the rebate increases with the barrier. In all, the contract value might decrease or increase in $\eta$ when the barrier is set higher. For the chosen parameters, the contract value and $\eta$ possess an opposite relation. Hence, for both the pension fund and the life insurance case, high $\delta$ ($\alpha \delta_L$) values shall be combined with high $\eta$ values to achieve a fair contract. The effects of $\eta$ and $\sigma$ on different contract components are summarized in Table 5.

<table>
<thead>
<tr>
<th></th>
<th>FP</th>
<th>DOC</th>
<th>DOP</th>
<th>RP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>↑</td>
<td></td>
<td></td>
<td>\↘</td>
</tr>
<tr>
<td>$\eta$</td>
<td></td>
<td></td>
<td></td>
<td>↑</td>
</tr>
</tbody>
</table>

Table 5. Effects of $\eta$, $\sigma$, and $L$ for different payoff components.

Additionally, in Figure 4, we illustrate the effect of indexation parameter $i$, which relates to the average CPI or wage growth rate over the life of the contract, on the fair participation rate. Since $\bar{L}$ is defined as the accrued value of $L$ with an accumulation factor of $i$, an increase in $i$ leads to an increases in $\bar{L}$. Recalling the payoff to the beneficiary (C.f. (1)), a higher $\bar{L}$ implies a lower value for the shorted call option, which means that $\delta$ should be set lower to achieve a fair contract. In total, a negative relation between $i$ and $\delta$ results
for the pension fund case. As \( i \) does not appear in the analysis of the life insurance case, it does not influence the fair participation rate.

4. Mean variance analysis

Through the analysis in Section 3, we notice the differences in the contract parameters (particularly the participation rate) a fair contract provides. Based on this, a naive annuity buyer might choose the contract providing the highest participation rate. Note that, until now, we have determined the fair participation rate by using the same investment policy or risk level (volatility) for both contracts. However, as already mentioned in the introduction, pension funds and life insurers are in practice differently exposed to risky assets. Hence, in what follows we want to make a comparative analysis to find out which financial institution is preferred over the other by a potential annuity buyer, provided that some differences, like regulatory constraints and volatility level, are taken into consideration. Since there is a one-to-one correspondence between volatility and default probability (c.f. Bernard and Chen (2009)), we assume that the pension fund and the life insurer will follow a risk management strategy with the maximum volatility level that satisfies the regulatory constraints. We use a confidence level of 97.5% for the pension fund and 99.9% for the life insurer. This allows the pension fund to opt for a more risky investment strategy. Although these confidence levels are specified on a one-year time horizon, they can also be translated to longer time horizons. The maximum allowed volatility level satisfying the
regulatory constraint is given by

$$\sigma^*(\eta, \varepsilon) = \arg\max \left\{ \sigma > 0 \left| P(\tau \leq T) = \varepsilon = 1 - \text{confidence level} \right. \right\}$$

with $\eta$ representing the minimum required funding level and $\varepsilon$ denoting the default probability constraint. The latter is of course 1 minus the regulatory confidence level.

In addition to the contract specifications, an annuity buyer also cares about the performance of the contract and related risks, i.e. the expected contract payoffs and the corresponding variances. Therefore, we assume that he will choose the contract that delivers the highest expected utility with respect to terminal wealth:

$$E[U(\text{terminal payment})] = U(P_0) + U'(P_0) E[\text{terminal payment} - P_0]$$

$$+ \frac{1}{2} U''(P_0) E[(\text{terminal payment} - P_0)^2]$$

where $P_0$ is the initial investment of the potential annuity buyer. We assume this to be the same in all cases. The terminal payment is the contract payoff at $T$ if there is no premature default or the rebate payment if premature default does occur. $U$ represents the utility function of the annuity buyer and it is assumed that $U' > 0$ and $U'' < 0$. Hereby we have assumed that the expected utility is merely dependent on the mean and the variance of the terminal wealth. It should be noted that the third or higher moments of terminal wealth are usually relevant to the utility analysis\textsuperscript{15}. However, the analysis is valid if the asset returns are normally distributed, which is the case in our context. Furthermore, the expectation is now taken under the real world measure $P$ rather than the equivalent martingale measure $P^*$. This is due to the fact that we are interested here in the performance of the contract and no longer in valuation. Under the real world measure $P$, the firm’s assets are assumed to evolve according to

$$dA_t = A_t(\mu dt + \sigma dW_t),$$

where $\mu$ and $\sigma$ are positive constants and $W_t$ is a Brownian motion under this measure. The expected utility of the terminal wealth from acquiring an annuity contract from the

\textsuperscript{15}Except for the quadratic utility where the third or higher moments are equal to zero
pension fund or the life insurance company is then determined by

\[
E[U(\text{payment from pension fund})] = U(P_0) + U'(P_0) E[\psi_B(A_T) 1_{\{\tau>T\}} + \Theta_B(\tau)e^{r(T-\tau)} 1_{\{\tau\leq T\}} - P_0]
\]

\[
+ 1/2U''(P_0) E[(\psi_B(A_T) 1_{\{\tau>T\}} + \Theta_B(\tau)e^{r(T-\tau)} 1_{\{\tau\leq T\}} - P_0)^2]
\]

(8)

\[
E[U(\text{payment from life insurance})] = U(P_0) + U'(P_0) E[\psi_L(A_T) 1_{\{\tau>T\}} + \Theta_L(\tau)e^{r(T-\tau)} 1_{\{\tau\leq T\}} - P_0]
\]

\[
+ 1/2U''(P_0) E[(\psi_L(A_T) 1_{\{\tau>T\}} + \Theta_L(\tau)e^{r(T-\tau)} 1_{\{\tau\leq T\}} - P_0)^2]
\]

(9)

For consistency reasons, we have here accumulated the rebate payment to maturity date \(T\) with risk-free interest rate \(r\).\(^{16}\) When the difference between these two expected utilities is positive, the contract provided by the pension fund is chosen over that provided by the life insurer. Otherwise the life insurer is preferred. To follow the above reasoning and to proceed our analysis, we shall now determine the default probabilities (to determine the maximum allowed volatility level) and the first and second moments of the terminal contract payoff.

4.1. Default Probability. Provided that the pension fund’s assets and those of the life insurer are assumed to follow the same stochastic evolution, the resulting default probability will be equal (for the same initial investment, barrier level and investment policy). The default probability is characterized by the probability that the firm’s assets have hit or fallen below the regulatory barrier before the maturity date, i.e.

\[
P(\tau \leq T) = P\left(\inf_{t \in [0,T]} \{A_t \leq B_t\} \leq T\right).
\]

(10)

In this framework, the default probability can be computed explicitly (c.f. Bernard and Chen (2009)):

\[
P(\tau \leq T) = N\left(d^+(B_0, A_0e^{(\mu-r)T})\right) + \left(\frac{A_0}{B_0}\right)^{-\frac{2\mu}{\sigma^2}} N\left(d^-(B_0, A_0e^{-(\mu-r)T})\right)
\]

(11)

\(^{16}\)We should keep in mind that different values of volatility will be used for calibration later.
with \( \hat{\mu} = \mu - r - \frac{\sigma^2}{2} \) and \( d^\pm(S, K) = \frac{\ln\left( \frac{S}{K} \right) \pm \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \). The maximum volatility level \( \sigma \) is determined numerically by making the following equation binding:

\[
P(\tau \leq T) = N\left(d^+(B_0, A_0 e^{(\mu - r)T})\right) + \left(\frac{A_0}{B_0}\right)^{\frac{-2\hat{\mu}}{\sigma^2}} N\left(d^-(B_0, A_0 e^{-(\mu - r)T})\right) = \varepsilon. \tag{12}
\]

For a one-year time horizon, the default probability constraint stipulated is 2.5% for pension funds and 0.5% for life insurance companies. In our analysis however we focus in the long term default probability. Table 6 illustrates several critical volatility levels for diverse values of the default probability constraint \( \varepsilon \) and the regulation parameter \( \eta \). For this, the default probability constraint has been adjusted to a 15-year time horizon.\(^{17}\) Since a high asset risk is usually accompanied by a higher expected rate of return, this also plays a role in the determination of the default probability. Here we assume a constant market price for risk (Sharpe ratio), i.e. we set \( \mu = r + \sigma \lambda \) with \( \lambda = 0.2 \) denoting the Sharpe ratio. Initially, a stricter regulation (a higher \( \eta \)) forces the life insurer or the pension fund to follow a more conservative risk management strategy, i.e. a low \( \sigma^\ast \) results for a given default probability constraint. Therefore a negative relation between \( \eta \) and \( \sigma \) is observed. For a given regulation parameter, say \( \eta = 1.05 \), a higher default probability constraint allows for a higher risk level. Table 7 shows the equivalent equity allocations from Table 6. These equity allocations have been derived assuming a diversified portfolio of equities and bonds using a volatility of equity returns of 18.0% and of bond returns of 6.0%. The correlation between equity and bond returns has been set at 0. Hereby we observe that for small maximally allowed default probability constraints, the equity allocations are relatively low, and the reversed effects hold for higher values of \( \varepsilon \). This result coincides with the observations given in Table 3.

4.2. **First and second moments of terminal payoff.** The determination of the expected utility difference between these two contracts boils down to deriving the first and second moments of terminal contract payoffs under the real world measure \( P \) weighted by \( U' \) and \( U'' \) of a certain utility function (C.f. (8) and (9)). Detailed expressions of these moments can be found in Appendix 5.2.

\(^{17}\)For instance, a default probability of 0.5% for a one-year time horizon implies a default probability of \( 1 - 0.995^{15} \approx 7.24\% \) for a 15-year time horizon.
Table 6. Critical volatility levels for different values of $\eta$ and $\varepsilon$ with parameters: $A_0 = 100; L = 120; T = 15; r = 0.05; \alpha = 0.9; \frac{\mu - r}{\sigma} = 0.2$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\varepsilon = 1%$</th>
<th>$\varepsilon = 2%$</th>
<th>$\varepsilon = 3%$</th>
<th>$\varepsilon = 4%$</th>
<th>$\varepsilon = 5%$</th>
<th>$\varepsilon = 6%$</th>
<th>$\varepsilon = 7%$</th>
<th>$\varepsilon = 8%$</th>
<th>$\varepsilon = 9%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta = 0.95$</td>
<td>0.0783</td>
<td>0.0878</td>
<td>0.0948</td>
<td>0.1007</td>
<td>0.1059</td>
<td>0.1107</td>
<td>0.1152</td>
<td>0.1195</td>
<td>0.1236</td>
</tr>
<tr>
<td>$\eta = 1.00$</td>
<td>0.0722</td>
<td>0.0809</td>
<td>0.0875</td>
<td>0.0929</td>
<td>0.0979</td>
<td>0.1024</td>
<td>0.1066</td>
<td>0.1106</td>
<td>0.1145</td>
</tr>
<tr>
<td>$\eta = 1.05$</td>
<td>0.0663</td>
<td>0.0744</td>
<td>0.0805</td>
<td>0.0856</td>
<td>0.0901</td>
<td>0.0943</td>
<td>0.0983</td>
<td>0.1020</td>
<td>0.1056</td>
</tr>
</tbody>
</table>

Table 7. Equity allocations for different values of $\eta$ and $\varepsilon$ with parameters: $A_0 = 100; L = 120; T = 15; r = 0.05; \alpha = 0.9; \frac{\mu - r}{\sigma} = 0.2$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\varepsilon = 1%$</th>
<th>$\varepsilon = 2%$</th>
<th>$\varepsilon = 3%$</th>
<th>$\varepsilon = 4%$</th>
<th>$\varepsilon = 5%$</th>
<th>$\varepsilon = 6%$</th>
<th>$\varepsilon = 7%$</th>
<th>$\varepsilon = 8%$</th>
<th>$\varepsilon = 9%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta = 0.95$</td>
<td>0.383</td>
<td>0.452</td>
<td>0.499</td>
<td>0.538</td>
<td>0.570</td>
<td>0.600</td>
<td>0.627</td>
<td>0.653</td>
<td>0.674</td>
</tr>
<tr>
<td>$\eta = 1.00$</td>
<td>0.334</td>
<td>0.403</td>
<td>0.450</td>
<td>0.488</td>
<td>0.520</td>
<td>0.550</td>
<td>0.577</td>
<td>0.603</td>
<td>0.624</td>
</tr>
<tr>
<td>$\eta = 1.05$</td>
<td>0.279</td>
<td>0.353</td>
<td>0.400</td>
<td>0.437</td>
<td>0.470</td>
<td>0.500</td>
<td>0.527</td>
<td>0.545</td>
<td>0.574</td>
</tr>
</tbody>
</table>

Figure 5. First moment $E[\tilde{V}_k(A_T)] = E[\psi_k(A_T)\mathbb{1}_{\{\tau > T\}} + \Theta_k(\tau)e^{(T-\tau)}\mathbb{1}_{\{\tau \leq T\}} - P_0], k = B, L$; “Pe” and “Li” stand for pension fund and life insurer, respectively.

Figures 5 and 6 respectively illustrate the first and the second moment of the terminal payoff for the life insurer and the pension fund as a function of the default probability.
Figure 6. Second moment $E[(\tilde{V}_k(A_T))^2] = E[\psi_k(A_T)1_{\{\tau>T\}} + \Theta_k(\tau) e^{r(T-\tau)} 1_{\{\tau\leq T\}} - P_0)^2]$, $k = B, L$; “Pe” and “Li” stand for pension fund and life insurer, respectively.

Constraint. To plot these graphs, the following steps have been taken. First, equivalent to Table 6 the maximum allowed volatility level $\sigma^*$ satisfying the regulatory constraints $(\eta, \varepsilon)$ is derived. Second, given this volatility level, the fair participation rate is computed. Third, given the volatility and the fair participation rate the first and second moment are calculated. Several observations are made based on these figures. The solid line represents the first (and second) moment for the life insurer. Both increase with the default probability constraint as higher volatility increases both expected return and variance. For equal default probability constraints (i.e. $\varepsilon_{Pe} = \varepsilon_{Li}$) both the first and second moment of the life insurer dominate over the pension fund (except for very high default probabilities), whereas when there is a less strict regulatory constraint for the pension fund both the first and second moments for the pension fund exceed those for the life insurer. To show this we allow the regulatory default parameter for a pension fund to be 4% higher than that of a life insurer, i.e. $\varepsilon_{Pe} = \varepsilon_{Li} + 4\%$. 19

Table 8 illustrates the expected utility difference for various utility functions. For each utility function we analyze three different probabilities of default. The default probability for the life insurer is set at 1%, 3% and 5%. For the pension fund we increase each of these default probabilities by 2.5% to 3.5%, 5.5% and 7.5%. This is consistent with the real life observation that the regulation of pension funds is less strict. This implies that

\[ \text{For ease of reference only the default probability constraint of the life insures had been shown on the x-axes.} \]

\[ \text{Hereby the choice of 4\% is quite arbitrary. More detailed effects of } \varepsilon \text{ are discussed later in this section.} \]
<table>
<thead>
<tr>
<th>Utility</th>
<th>$\varepsilon_{Li}$</th>
<th>$EU_{Diff}$</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(x) = x$</td>
<td>0.01</td>
<td>-2.790</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>-4.322</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>-3.700</td>
<td>Life</td>
</tr>
<tr>
<td>$U(x) = x - 0.0001x^2$</td>
<td>0.01</td>
<td>-2.728</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>-4.144</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>-3.541</td>
<td>Life</td>
</tr>
<tr>
<td>$U(x) = 1 - e^{-\gamma x}$</td>
<td>0.01</td>
<td>0</td>
<td>Indifferent</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0</td>
<td>Indifferent</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0</td>
<td>Indifferent</td>
</tr>
<tr>
<td>$U(x) = \log x$</td>
<td>0.01</td>
<td>-0.024</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0.014</td>
<td>Pension</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.016</td>
<td>Pension</td>
</tr>
<tr>
<td>$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma = 0.8$</td>
<td>0.01</td>
<td>-0.062</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0.004</td>
<td>Pension</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.011</td>
<td>Pension</td>
</tr>
<tr>
<td>$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma = 0.9$</td>
<td>0.01</td>
<td>-0.039</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0.0126</td>
<td>Pension</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.0156</td>
<td>Pension</td>
</tr>
<tr>
<td>$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma = 1.2$</td>
<td>0.01</td>
<td>-0.090</td>
<td>Life</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0.0108</td>
<td>Pension</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.0109</td>
<td>Pension</td>
</tr>
</tbody>
</table>

Table 8. $EU_{Diff} := E[U(Pension)] - E[U(Life)]$ with parameters: $A_0 = 100; L = 120; T = 15; r = 0.05; i = 0.025; \alpha = 0.9; \frac{\mu-r}{\sigma} = 0.2$. $\varepsilon_P$ is set 2.5% higher than $\varepsilon_{Li}$. The pension fund can choose a more risky investment strategy. In case of linear utility, the pivotal element is the first moment $E[\tilde{V}_k(A_T)]$, $k = B, L$. For the given parameters, a linear-annuity buyer who only cares about the expected terminal payment would always choose a life insurer over a pension fund. The same result is observed for an annuity buyer with quadratic utility. The reason behind this, is that the second derivative of the
quadratic utility is very small, which implies that the decision mainly depends on the ex-
pected terminal payment. It is interesting to observe that an exponential-annuity buyer
is indifferent with respect to the choice between a pension fund and a life insurer. For the
utility functions mentioned above, the default probability constraint does not seem to play
an important role. When it comes to log and power utilities, both the life insurer and the
pension fund can be chosen, depending on the level of the default probability constraint.
For a low level of default probability, the life insurer is preferred over a pension fund.

This analysis shows that default probability can play a critical role in the annuity-
purchasing decision of a consumer, depending on his preferences. Taking power utility as
an example, with the risk aversion parameter $\gamma$ equal to 0.8 (c.f. Figure 7), we observe
that the expected utility from purchasing an annuity from a life insurer is always smaller,
if we neglect the regulatory differences for the life insurance and pension fund. While the
decision can be reversed when the regulator is less strict with the pension fund. Natu-
rally, the question arises what the critical value of $\varepsilon_{Pe} - \varepsilon_{Li}$ is, such that the individual
is indifferent in his choice between investing in a life insurer or pension fund. Figure 8
and Table 9 report some answers to this question. In Figure 8, hump-shaped curves are
observed for different degrees of relative risk aversion $\gamma$. This implies that as a higher
default probability constraint $\varepsilon_{Li}$ is allowed, a higher difference $\varepsilon_{Pe} - \varepsilon_{Li}$ should be pro-
vided such that indifference between these two institutions can be reached. After a certain
level of $\varepsilon_{Li}$, the regulatory difference can be lowered again to provide the indifferent choice.

In Table 9, some critical values for this difference are calculated for different levels of $\gamma$
and $\varepsilon_{Li}$. For a risk aversion parameter $\gamma = 0$ the difference between the default probability
of the life insurer and the pension fund should be between 3.3 and 4.0%. For a risk aversion
parameter $\gamma = 0.8$ the default probability of the pension fund should be 2 to 2.5% higher
to make the consumer indifferent and for $\gamma = 1.2$ the difference ranges from 2.3 to 2.8%.
Figure 7. Expected power utility with $\gamma = 0.8$ for life insurance and pension funds (with “Pe” and “Li” standing for pension fund and life insurer).

Figure 8. Differences $\varepsilon_{Pe} - \varepsilon_{Li}$ for indifferent power annuity buyer with different levels of risk aversion $\gamma$.

5. Conclusion

The present paper analyzes the institutional differences between defined benefit pension funds and life insurance companies, which are both important annuity providers. The residual risk in a (non-profit) pension fund is collectively borne by the beneficiaries and the corporate sponsor. In case of an insurance company, the residual risk is borne by the external shareholders. First, we determine fair combinations of contract parameters by applying a contingent claim approach and compare the fair participation rates in relation to the investment policy in both contracts. We find that for realistic parameters the fair participation rate in case of the pension fund is typically lower than the equivalent rate in case of the life insurer. Second, by using mean-variance analysis we answer the question, which contract an annuity buyer would prefer based on expected utility of terminal wealth. This part of the analysis is performed by incorporating the regulatory requirements that
power utility \( \gamma = 0 \) power utility \( \gamma = 0.8 \) power utility \( \gamma = 1.2 \)

<table>
<thead>
<tr>
<th>( \varepsilon_{Li} )</th>
<th>( \varepsilon^*_{Pe} )</th>
<th>( \varepsilon^*<em>{Pe} - \varepsilon</em>{Li} )</th>
<th>( \varepsilon^*_{Pe} )</th>
<th>( \varepsilon^*<em>{Pe} - \varepsilon</em>{Li} )</th>
<th>( \varepsilon^*_{Pe} )</th>
<th>( \varepsilon^*<em>{Pe} - \varepsilon</em>{Li} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>4.34%</td>
<td>3.34%</td>
<td>3.09%</td>
<td>2.09%</td>
<td>3.30%</td>
<td>2.30%</td>
</tr>
<tr>
<td>2%</td>
<td>5.86%</td>
<td>3.86%</td>
<td>4.40%</td>
<td>2.40%</td>
<td>4.62%</td>
<td>2.62%</td>
</tr>
<tr>
<td>3%</td>
<td>7.04%</td>
<td>4.04%</td>
<td>5.53%</td>
<td>2.53%</td>
<td>5.74%</td>
<td>2.74%</td>
</tr>
<tr>
<td>4%</td>
<td>8.04%</td>
<td>4.04%</td>
<td>6.57%</td>
<td>2.57%</td>
<td>6.76%</td>
<td>2.76%</td>
</tr>
<tr>
<td>5%</td>
<td>8.93%</td>
<td>3.93%</td>
<td>7.56%</td>
<td>2.56%</td>
<td>7.73%</td>
<td>2.73%</td>
</tr>
<tr>
<td>6%</td>
<td>9.75%</td>
<td>3.75%</td>
<td>8.53%</td>
<td>2.53%</td>
<td>8.67%</td>
<td>2.67%</td>
</tr>
<tr>
<td>7%</td>
<td>10.52%</td>
<td>3.52%</td>
<td>9.47%</td>
<td>2.47%</td>
<td>9.59%</td>
<td>2.59%</td>
</tr>
</tbody>
</table>

Table 9. Indifference combinations of \( \varepsilon_{Li} \) and \( \varepsilon_{Pe} \) for power utility \( U(x) = \frac{x^{1-\gamma}}{1-\gamma} \) with different degrees of relative risk aversion: \( \gamma = 0 \) (linear case), \( \gamma = 0.8 \), \( \gamma = 1.2 \).

determine the maximum allowed asset volatility of the risk management strategies the life insurer or pension fund can trade in. We observe that the annuity buying choice of a consumer crucially depends on his preferences. Under linear and quadratic utility, life insurance companies dominate pension funds. For power utility functions the choice between a pension fund or a life insurer depends on the level of regulatory default probability. Assuming log utility both institutions can be chosen. And finally, assuming exponential utility, the consumer is indifferent to a choice between a pension fund or a life insurance company. These insights are relevant for many practical questions, e.g. for optimal pension plan design and fair contract arrangements. Furthermore, we find that power utility consumers are indifferent if the long term regulatory default probability of pension funds exceeds that of life insurers by 2 to 4%. This supports differences in regulatory regimes found in reality.

**Appendices**

5.1. **Appendix A.** The value of the fixed payment \( FP(L) \) is equal to

\[
E^*[e^{-rT}L_{1_{\{r>T\}}}] = Le^{-rT} \left[ N(d^- (A_0, B_0)) - \left( \frac{A_0}{B_0} \right) N(d^- (B_0, A_0)) \right]
\]
with \( d^\pm(S, K) = \frac{\ln(\frac{S}{K}) \pm \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \) and \( B_0 = L e^{-r T} \). The value of the down-and-out call option with strike \( K, K = L, L/\alpha, \bar{L} \) is given by

\[
E^*[e^{-r T}[A_T - K]^+1_{\{\tau>T\}}] \\
= A_0 N(d^+(A_0, K e^{-r T})) - K e^{-r T} N \left( d^- \left( A_0, K e^{-r T} \right) \right) \\
- \left( \frac{A_0}{B_0} \right) \left( B_0^2 \right) N \left( d^+ \left( \frac{B_0^2}{A_0}, \max\{K e^{-r T}, B_0\} \right) \right) \left( d^- \left( \frac{B_0^2}{A_0}, \max\{K e^{-r T}, B_0\} \right) \right)
\]

The value of the down-and-out put option \( DOP(L) \) can be calculated by

\[
E^*[e^{-r T}[L - A_T]^+1_{\{\tau>T\}}] \\
= 1_{\{\eta<1\}} e^{-r T} L \left[ N(-d^- (A_0, L e^{-r T})) - N(d^+(B_0, A_0)) - \frac{A_0}{B_0} N(d^- (B_0, A_0)) + \frac{A_0}{B_0} N(d^+(B_0, L e^{-r T})) \right] \\
- A_0 \left[ N(-d^+(A_0, L e^{-r T})) - N(d^+(A_0, B_0)) - \frac{B_0}{A_0} N(d^+(B_0, A_0)) + \frac{B_0}{A_0} N(d^+(B_0, L e^{-r T})) \right]
\]

And the value of the rebate payment \( RP_B = RP_L \) is determined by

\[
E^*[e^{-r T} \min\{\eta, 1\} L e^{-r(T-\tau)} 1_{\{\tau \leq T\}}] \\
= \min\{\eta, 1\} L e^{-r T} \left( N(-d^- (A_0, B_0)) + \left( \frac{A_0}{B_0} \right) N(d^- (B_0, A_0)) \right).
\]

The value of rebate payment \( RP_S \) and \( RP_E \) can be calculated analogously.

5.2. Appendix B. Since the expected final payment can be derived quite similarly as for the valuation part, we will jump to the results directly. For \( K = L, \frac{L}{\alpha}, L e^{i T} \), it holds

\[
E[[A_T - K]^+1_{\{\tau>T\}}] = A_0 e^{\mu T} N \left( d^+(A_0, X e^{-\mu T}) \right) - K N \left( d^- (A_0, X e^{-\mu T}) \right) \\
- A_0 e^{\mu T} \left( \frac{B_0}{A_0} \right) \frac{2(\mu - r + \frac{1}{2} \sigma^2)}{\sigma^2} \left( N \left( d^+ ((B_0)^2, A_0 X e^{-\mu T}) \right) \right) \\
+ K \left( \frac{B_0}{A_0} \right) \frac{2(\mu - r + \frac{1}{2} \sigma^2)}{\sigma^2} \left( N \left( d^- ((B_0)^2, A_0 X e^{-\mu T}) \right) \right)
\]

with \( X := \max\{B_T, K\} \).

The fixed payment is determined by

\[
E[L 1_{\{\tau>T\}}] = L N \left( d^- (A_0, B_0 e^{-(\mu-r) T}) \right) - L \left( \frac{A_0}{B_0} \right) \frac{-2(\mu - r + \frac{1}{2} \sigma^2)}{\sigma^2} \left( N \left( d^- (B_0, A_0 e^{-(\mu-r) T}) \right) \right).
\]
The down–and–out put option is then determined by

\[
E[L - A_T]^+ 1_{\{\tau > T\}}
\]

\[
= L \left[ N(-d^-(A_0, Le^{-\mu T})) - N(d^+(B_0, A_0e^{(\mu-r)T})) - \left( \frac{B_0}{A_0} \right)^{\frac{1}{2} + \frac{1}{2} \sigma^2} N(d^-(B_0, A_0e^{(\mu-r)T})) \right.
\]

\[
+ \left( \frac{B_0}{A_0} \right)^{\frac{1}{2} + \frac{1}{2} \sigma^2} N(d^+(B_0, A_0e^{(r-\mu)T})) + \left( \frac{B_0}{A_0} \right)^{\frac{1}{2} + \frac{1}{2} \sigma^2} N(d^+(B_0, A_0Le^{-\mu T})) \right]
\]

In order to compute the second moments of the terminal payments, we shall calculate several expectations which take the form of

\[
E \left[ (\ln A_T - K)^+ 1_{\{\tau > T\}} \right]^2
\]

\[
= E[A_T^2 1_{\{A_T > K, \tau > T\}}] - 2K E[A_T 1_{\{A_T > K, \tau > T\}}] + K^2 E[1_{\{A_T > K, \tau > T\}}]
\]

\[
= \left( E[A_T^2 1_{\{A_T > K\}}] - E[A_T^2 1_{\{A_T > K, \tau \leq T\}}] \right) - \left( 2K E[A_T 1_{\{A_T > K\}}] - 2K E[A_T 1_{\{A_T > K, \tau \leq T\}}] \right)
\]

\[
+ \left( K^2 E[1_{\{A_T > K\}}] - K^2 E[1_{\{A_T > K, \tau \leq T\}}] \right).
\]

Using the following notations:

\[
b := \frac{1}{\sigma} \ln \frac{B_0}{A_0}; \quad m = \frac{\mu - r - 1/2 \sigma^2}{\sigma}, \quad k = \frac{1}{\sigma} \left( \ln K - rT \right),
\]

all the six terms in (13) are explicitly given by

\[
E[A_T^2 1_{\{A_T > K\}}] = A_0^2 e^{2\mu + \sigma^2 T} N \left( \frac{\ln A_0 + (\mu + \frac{3}{2} \sigma^2) T}{\sigma \sqrt{T}} \right)
\]

\[
E[A_T^2 1_{\{A_T > K, \tau \leq T\}}] = A_0^2 e^{2\mu T} e^{2b(m+2\sigma)e^{(2m\sigma^2T+2\sigma^2T)}N} \left( \frac{2b - k + (m + 2\sigma)T}{\sigma \sqrt{T}} \right)
\]

\[
2K E[A_T 1_{\{A_T > K\}}] = 2K A_0 e^{\mu T} N \left( \frac{\ln A_0 + (\mu + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right)
\]

\[
2K E[A_T 1_{\{A_T > K, \tau \leq T\}}] = 2K A_0 e^{\tau T} e^{2b(m+\sigma)e^{m\sigma T}e^{\frac{1}{2} \sigma^2 T}N} \left( \frac{2b - k + (m + \sigma)T}{\sigma \sqrt{T}} \right)
\]

\[
K^2 E[1_{\{A_T > K\}}] = K^2 N \left( \frac{\ln A_0 + (\mu - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right)
\]

\[
K^2 E[1_{\{A_T > K, \tau \leq T\}}] = K^2 e^{2bm N} \left( \frac{2b - k + mT}{\sigma \sqrt{T}} \right).
\]

28
When coming to compute the second moment of the down-and-out put, we rely on the following relations:

\[
E[(L - A_T)^2 1_{\{\tau > T\}}]
= E[(L - A_T)^2 1_{\{\tau > T, A_T < L\}}]
= E[L^2 1_{\{\tau > T, A_T < L\}}] - 2LE[AT 1_{\{\tau > T, A_T < L\}}] + E[A_T^2 1_{\{\tau > T, A_T < L\}}]
= \left( E[L^2 1_{\{A_T < L\}}] - E[L^2 1_{\{\tau \leq T\}}] + E[L^2 1_{\{\tau \leq T, A_T > L\}}]\right)
- 2L \left( E[A_T 1_{\{A_T < L\}}] - (E[A_T 1_{\{\tau \leq T, A_T > B_T\}}] + E[A_T 1_{\{A_T \leq B_T\}}]) + E[A_T 1_{\{\tau \leq T, A_T > L\}}]\right)
+ \left( E[A_T^2 1_{\{A_T < L\}}] - ((E[A_T^2 1_{\{\tau \leq T, A_T > B_T\}}] + E[A_T^2 1_{\{A_T \leq B_T\}}]) + E[A_T^2 1_{\{\tau \leq T, A_T > L\}}]\right).
\]

Furthermore, we shall calculate some terms which have the following forms:

\[
E [(A_T - K_1)^+ (A_T - K_2)^+ 1_{\{\tau > T\}}]
= E[(A_T - K_1)(A_T - K_2) 1_{\{A_T > \max\{K_1, K_2\}, \tau > T\}}]
= E[\left(A_T^2 - (K_1 + K_2) A_T + K_1 K_2\right) 1_{\{A_T > \max\{K_1, K_2\}, \tau > T\}}]
= \left(E[A_T^2 1_{\{A_T > \max\{K_1, K_2\}\}}] - E[A_T^2 1_{\{A_T > \max\{K_1, K_2\}, \tau \leq T\}}]\right)
- (K_1 + K_2) \left(E[A_T 1_{\{A_T > \max\{K_1, K_2\}\}}] - E[A_T 1_{\{A_T > \max\{K_1, K_2\}, \tau \leq T\}}]\right)
+ K_1 K_2 \left(E[1_{\{A_T > K\}}] - E[1_{\{A_T > K, \tau \leq T\}}]\right)
\]

Finally, it should be noted that

\[
E \left[\left(A_T - \frac{L}{\alpha}\right)^+ [L - A_T]^+ 1_{\{\tau > T\}}\right] = 0.
\]
References


| No. 198 | Peter ter Berg, Unification of the Fréchet and Weibull Distribution |
| No. 199 | Ronald Heijmans, Simulations in the Dutch interbank payment system: A sensitivity analysis |
| No. 200 | Itai Agur, What Institutional Structure for the Lender of Last Resort? |
| No. 201 | Iman van Lelyveld, Franka Liedorp and Manuel Kampman, An empirical assessment of reinsurance risk |
| No. 202 | Kerstin Bernoth and Andreas Pick, Forecasting the fragility of the banking and insurance sector |
| No. 203 | Maria Demertzis, The ‘Wisdom of the Crowds’ and Public Policy |
| No. 204 | Wouter den Haan and Vincent Sterk, The comovement between household loans and real activity |
| No. 205 | Gus Garita and Chen Zhou, Can Open Capital Markets Help Avoid Currency Crises? |
| No. 206 | Frederik van der Ploeg and Steven Poelhekke, The Volatility Curse: Revisiting the Paradox of Plenty |
| No. 207 | M. Hashem Pesaran and Adreas Pick, Forecasting Random Walks under Drift Instability |
| No. 208 | Zsolt Darvas, Monetary Transmission in three Central European Economies: Evidence from Time-Varying Coefficient Vector Autoregressions |
| No. 209 | Steven Poelhekke, Human Capital and Employment Growth in German Metropolitan Areas: New Evidence |
| No. 210 | Vincent Sterk, Credit Frictions and the Comovement between Durable and Non-durable Consumption |
| No. 211 | Jan de Dreu and Jacob Bikker, Pension fund sophistication and investment policy |
| No. 212 | Jakob de Haan and David-Jan Jansen, The communication policy of the European Central Bank: An overview of the first decade |
| No. 213 | Itai Agur, Regulatory Competition and Bank Risk Taking |
| No. 214 | John Lewis, Fiscal policy in Central and Eastern Europe with real time data: Cyclicality, inertia and the role of EU accession |
| No. 215 | Jacob Bikker, An extended gravity model with substitution applied to international trade |
| No. 216 | Arie Kapteyn and Federica Teppa, Subjective Measures of Risk Aversion, Fixed Costs, and Portfolio Choice |
| No. 217 | Mark Mink and Jochen Mierau, Measuring Stock Market Contagion with an Application to the Sub-prime Crisis |
| No. 218 | Michael Biggs, Thomas Mayer and Andreas Pick, Credit and economic recovery |
| No. 219 | Chen Zhou, Dependence structure of risk factors and diversification effects |
| No. 220 | W.L. Heeringa and A.L. Bovenberg, Stabilizing pay-as-you-go pension schemes in the face of rising longevity and falling fertility: an application to the Netherlands |
| No. 221 | Nicole Jonker and Anneke Kosse, The impact of survey design on research outcomes: A case study of seven pilots measuring cash usage in the Netherlands |
| No. 222 | Gabriele Galati, Steven Poelhekke and Chen Zhou, Did the crisis affect inflation expectations? |
| No. 223 | Jacob Bikker, Dirk Broeders, David Hollanders and Eduard Ponds, Pension funds’ asset allocation and participants age: a test of the life-cycle model |
| No. 224 | Stijn Claessens and Neeltje van Horen, Being a Foreigner among Domestic Banks: Asset or Liability? |
| No. 225 | Jacob Bikker, Sherrill Shaffer and Laura Spierdijk, Assessing Competition with the Panzar-Rosse Model: The Role of Scale, Costs, and Equilibrium |
| No. 226 | Jan Marc Berk and Beata K. Bierut, Communication in a monetary policy committee: a note |