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Dirk Broeders

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D.W.G.A. Broeders*
De Nederlandsche Bank
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Abstract

This paper analyzes the relationship between a pension fund with conditionally indexed defined benefit liabilities and its sponsor, using contingent claims analysis. As pension funds run a mismatch risk, future surpluses and shortfalls will occur. Surpluses are divided between beneficiaries and sponsor through conditional indexation and refunding. Covering an eventual loss at the pension fund level is a function of the sponsor’s financial ability to do so. This paper suggests that this system creates an asymmetric allocation of the residual risk between sponsor and beneficiaries. The main conclusion is that the sponsor’s vulnerability negatively impacts the optimum risk profile of a defined benefit scheme with conditional indexation and thereby the market value of conditional indexation.

JEL Classification: G11; G23

Keywords: Defined benefit, pension put, conditional indexation, vulnerable options

*De Nederlandsche Bank (DNB), Supervisory Policy Division, Strategy Department, PO Box 98, 1000 AB Amsterdam. E-mail address: d.w.g.a.broeders@dnb.nl; Fax: +31 20 524 1885; Tel.: +31 20 524 5794. The views expressed in this paper are those of the author and not necessarily those of DNB. The author is grateful to Jaap Bikker, Zvi Bodie, Aerdt Houben, Jan Kakes, Theo Nijman, Eduard Ponds, Marc Pröpper, Gaston Siegelaer, Jonathan Treussard and Peter Vlaar for valuable suggestions and discussions.
1 Introduction

In its principles for the regulation of occupational pension schemes the OECD states that pension funds must be legally separated from the sponsor. This detachment is also prescribed by the EU’s European Pension Directive. Pension funds thus serve as special purpose vehicles to ensure that the accrued pension rights of the beneficiaries are not subjected to the sponsor’s default risk. In reality, however, there is no clean break between the sponsor and its pension fund. Particularly in the case of funded defined benefit schemes, subsequent situations of overfunding and underfunding may lead to additional cash flows between the two parties. Risk management at the pension fund level largely accounts for this. Pension funds in general do not, or are unable to, invest in the portfolio that exactly replicates the size and nature of their liabilities. Generally there is a lack of suitable guaranteed real return investment opportunities. Under this restriction, providing unconditionally inflation or wage indexed pensions might become infeasible at reasonable costs. Therefore, in many defined benefit pension deals, part of the pension promise is contingent on the performance of the pension fund assets. In return for taking mismatch risk pension fund trustees accept the possibility of encountering strong or weak financial conditions. This not only affects the pension fund and its beneficiaries, but also the sponsor. It is well-known in the literature that the funding status of the defined benefit pension plan is reflected in the market value of the sponsor, even if the pension fund is legally separated. For an overview see Jin et al (2004). In addition to the ‘value transparency’ argument, Jin et al (2004) find that the market risk of the sponsor’s equity reflects the risk level of the pension plan. In case of a funding deficit at the pension fund level, the sponsor may have the legal or moral obligation to increase contributions to the fund. On the other hand, surpluses in the pension fund tend to be claimed by the sponsor. Contribution holidays are a common phenomenon. Through these implicit contingent claims on the pension fund’s assets, there is a distinct relation between the pension fund and its sponsor.

This paper investigates the impact of these implicit contingent claims on optimal pension fund risk management. Given the potential distribution of future losses and surpluses, questions are what the optimal investment policy of the fund is and what the market value of the conditional indexation clause in the pension deal is. This is relevant for individuals since they need the value and the riskiness of their defined benefit pension savings for optimal life-cycle planning. Also the paper provides tools for evaluating and optimizing pension fund risk management through incorporating full disclo-
sure of indexation rules and sponsor dependence. The paper is structured as follows. Section 2 reviews preliminary concepts relevant for defined benefit pension fund risk management and valuation practices. Section 3 reviews the classical pension put as introduced by Treynor (1977). Subsequent sections consider the liability valuation problem in relation to sponsor risk from the pension fund’s perspective. Following an outline of a general framework of conditional liabilities in section 4, section 5 assumes that the sponsor unconditionally clears all investment losses within the pension fund. Sections 6 relaxes this assumption to the extent that the sponsor offers a limited guarantee or, alternatively, a partial loss insurance. The next step in section 7 is to include sponsor specific characteristics, specifically the financial ability to back the pension promises. This may be modelled as a vulnerable put option. Section 8 translates the risk profile into portfolio weights. The final two sections consider possible extensions of the analysis and summarize the paper. The appendices explain the technical details.

2 Environment and preliminary concepts

This section reviews some general remarks on risk management and valuation for a pension fund with defined benefit liabilities. The starting assumption is that asset prices follow a geometric Brownian motion:

\[ dS = \mu S dt + \sigma S d\omega \]  \hspace{1cm} (1)

with \( \mu \) the constant expected return per unit of time, \( \sigma^2 \) the constant variance of returns per unit of time and \( S \) the market value of the pension fund assets at time \( t \). The time subscript is suppressed for ease of notation. The source of uncertainty is a Wiener process \( \omega \). The distribution of the market value of the assets at maturity, \( S_T \), is lognormal, the continuously compounded return between \( t = 0 \) and \( t = T \) is normally distributed. Using Itô’s lemma, see Hull (2003), this implies that the change in the portfolio’s value over time is:

\[ \ln(\frac{S_T}{S}) \sim \Phi \left( \frac{1}{2} \sigma^2 (T - t), \sigma \sqrt{T - t} \right) \]  \hspace{1cm} (2)

where \( \Phi \) represents the normal distribution. In case of a defined benefit, a payment \( L_n \) is guaranteed to the beneficiaries at maturity \( t = T \). So, the market value of the pension fund’s assets at maturity must be at least equal to \( L_n \). A case of default is defined as a situation in which the pension fund has insufficient assets to pay the beneficiaries in full at maturity \( (S_T < L_n) \).
Over the duration of the pension deal the pension fund trustees have to manage the assets in relation to their liabilities. Two related measures are important in managing the shortfall risk: the probability of a loss and its expected value. In Appendix A.1 it is shown that the probability of default (PD) equals

\[ PD = P(S_T < L_n) = N(-d_2) \]  

(3)

with \( N \) the cumulative normal distribution function and parameter \( d_2 \) equal to

\[ d_2 = \frac{\ln(S/L_n) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \]  

(4)

where bold face distinguishes this parameter from the equivalent risk-neutral parameter of the Black-Scholes-Merton framework. Figure (1) plots the probability of default as a function of the time to maturity for different initial funding ratios. The funding ratio is the market value of the assets divided by the market value of the liabilities discounted at the risk-free rate:

\[ F = S/(L_n e^{-r_F(T-t)}) \]

Starting from a situation of overfunding \((F > 1)\), the probability of underfunding \( P(S_T < L_n) \) initially increases with time to maturity. For instance, starting with a funding ratio of 130%, an annual expected return on assets of 8% with volatility 14.14% and a risk-free return of 5%, returns a probability of default on a one year horizon of 2.3%. Evaluated on a 5 year horizon this is 13.0%. This, however, is not a general result. After a certain time to maturity the default probability starts to decrease. From the derivation in Appendix A.1 the time to maturity at which the underfunding probability is at maximum value, given an initial funding ratio in excess of 100%, is given by

\[ T - t^* = \ln(F)/\left(\mu - r_F - \frac{1}{2}\sigma^2\right) \]  

(5)

For instance, in the numerical example the probability of underfunding is the highest for a holding period of approximately 13 years. For longer maturities this risk measure decreases. This invalidates the measure for long term risk management and life-cycle planning, see Treussard (2005) for extensive considerations on this. In addition, from equation (5) it follows directly that a the turning point \((T - t^*)\) is highly sensitive to the expected risk premium \((\mu - r_F)\). In the example the turning point doubles to 26 years if the expected risk premium is lowered by 1 percentage point. For initial funding ratios at or below 100% figure (1) shows that the default probability
Figure 1: Probability of default (PD) in equation (3) for different initial funding ratios, using $r_F = 0.05$, $\mu = 0.08$, $\sigma^2 = 0.02$ and $L_n = 100e^{r_F(T-t)}$.

is monotonic decreasing for maturities, however the absolute level is higher compared to initial funding ratios in excess of 100%.

A shortcoming of the probability of default is that it is a one-dimensional measure of risk. It does not take into account the severity of the shortfall. This aspect, however, can be quantified using another risk measure: loss given default (LGD). The LGD is defined as the conditional expectation of the market value of the pension fund’s assets at maturity, given that these are less than the guaranteed pension benefit, or formally

$$LGD = E(S_T|S_T < L_n) = \frac{\int_0^{L_n} S_T f(S_T) dS_T}{P(S_T < L_n)}$$  \hspace{1cm} (6)

From Appendix A.1 the solution to equation (6) can be written as

$$E(S_T|S_T < L_n) = Se^{\mu(T-t)}N(-d_1)/N(-d_2)$$  \hspace{1cm} (7)

with parameter $d_2$ in equation (4) and parameter $d_1$ defined as

$$d_1 = d_2 + \sigma \sqrt{T-t}$$  \hspace{1cm} (8)

Figure (2) plots the present value of the loss given default, or $LGD e^{-r_F(T-t)}$, for different initial funding ratios and maturities up to 100 years. A higher
initial funding ratio lowers the expected value of the shortfall. However, for any initial funding ratio the present value of the loss is a monotonic increasing function of the time to maturity. The present value is taken for comparability of the loss given default over different horizons.

Both dimensions of risk (probability and severity) are taking into account in option pricing, as shown in the next section. Option pricing for this reason may be a useful tool for pension fund risk management. Also, contingent claims analysis can be used to show that in a complete market a pension fund is a zero sum game amongst the relevant parties: corporations, corporate stockholders and employees. If everything is traded and all parties have recourse to the capital market, one can always take positions that will complement or offset those resulting from corporate funding decisions, Treynor (1977). This however does not imply that collective pension funds have no reason to exist. One simple argument in favor is that markets are not complete and not all participants have access to capital markets. This implies e.g. that pooling longevity risk is a Pareto welfare improvement for the individual participants. Cui et al (2005) describe pension funds as a potentially positive-sum game in welfare-terms. Pension deals that provide
safer and smoother consumption streams will increase the utility of the parties involved. This welfare-enhancing aspect comes from intergenerational risk sharing.

Another central theme in pension finance is valuation. The prevailing concept is that the market value of pension liabilities is determined by the nature of the pension claim. This leads to the replication principle: the current value of a future pension benefit is the market price of a portfolio strategy that replicates the promised payoffs in all future states of the world. For instance, expected cash flows from a nominal defined benefit promise can be replicated with nominal zero-coupon bonds with very low default risk. This is equivalent to discounting the cash flows at the prevailing nominal risk-free spot rate. Unconditionally indexed pension liabilities are discounted at the real interest rate derived from index linked bonds. This, however, implies a complete market. De Jong (2005) discusses several methods to value pension liabilities in incomplete markets, specifically for wage indexed liabilities. De Jong advocates a utility-based valuation framework which corrects for expected real wage growth and real wage risk. Valuation will be explored further in the next sections. To start the analysis it is assumed in the next section that future pension payments are ‘on balance’ liabilities of the sponsoring company. In the subsequent sections a separated and capitalized pension fund is introduced.

3 Pension liabilities as corporate debt

In absence of a legally separated and fully funded pension scheme, corporate finance regards pension liabilities as (senior) corporate debt for the company. In order to fully honor the pension rights without residual risk, the assets backing this pension promise should be invested in assets exactly replicating the cash flows pattern. This investment strategy is usually not followed in corporate pension funding decisions. As a result, default risk is introduced with respect to the fulfilment of the defined benefit claim. Treynor (1977) describes this as the ‘pension put,’ or a sponsor’s claim against the pension beneficiaries representing an asset of the sponsor. In that context, the net value of the pension claim is the contractual cash flow, discounted at a risk-free rate, minus the market value of the pension put. The value of the pension put is the compensation pension fund participants claim for running default risk. The pension put can also be interpreted as the market value of the insurance contract that guarantees that all pension obligations will be met. Assuming rational behavior, the representatives of employers and
employees responsible for the total wage bill include the value of the pension put in their negotiations. A higher value of the pension put should be compensated by a higher current wage for the employees. Again, assuming rational behavior and a complete market, beneficiaries negotiate and should receive exactly enough money to be able to engage an offsetting transaction in the capital market.

To do a formal analysis an elementary case is presented in which the pension promise is a single nominal cash flow of \( L_n \) at time \( t = T \) to a cohort of beneficiaries. One can also think of \( L_n \) representing a sequence of cash flows with an equivalent duration. The defined benefit is usually related to final or average pay and years of service. \( L_n \) includes mortality and longevity risk over the time to maturity of the contract. The present value of the cash flow equals the difference between the market price of a default-free zero-coupon bond \( B \) paying \( L_n \) at maturity and the pension put \( P_{L_n} \).

The market value of the pension put at \( t = T \) is determined by the exercise price (\( L_n \)), the time to maturity (\( T - t \)) and accompanying risk-free spot rate (\( r_F \)), the market value and the risk profile of the assets (\( S \)) backing the pension liability. The present value of the pension liability equals

\[
P V(L_n) = B - P_{L_n} \tag{9}
\]

Since there are no intermediate cash flows, the option can only be exercised at maturity and the asset volatility is assumed to be constant, the standard Black-Scholes-Merton framework can be used for evaluating the market value of the pension put. In doing so the implied discount rate \( z \), taking into account the default risk, can be derived from

\[
L_n e^{-z(T-t)} = L_n e^{-r_F(T-t)} - \left\{ L_n e^{-r_F(T-t)} N(-d_2) - SN(-d_1) \right\} \tag{10}
\]

where \( N() \) is the cumulative probability distribution for a standard normal variable. The familiar option pricing parameters from Black & Scholes (1973) are

\[
d_1 = \frac{\ln(S/L_n) + (r_F + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \tag{11}
\]

The market value of the zero-coupon bond equals \( B = L_n e^{-r_F(T-t)} \). Rewriting equation (10), applying the property of the normal distribution that \( N(x) = 1 - N(-x) \), leads to the following expression for the option-adjusted
discount rate
\[
z = r_F - (T - t)^{-1} \ln \left\{ N(d_2) + \frac{S}{B} N(-d_1) \right\}
\] (12)

Note that \( z \geq r_F \) as \( 0 < N(d_2) + \frac{S}{B} N(-d_1) < 1 \). This statement is
demonstrated by the limits: \( S \to 0 : N(d_2) \to 0 \wedge N(-d_1) \to 1 \) and
\( S \to \infty : N(d_2) \to 1 \wedge N(-d_1) \to 0 \). For the special case \( S = B \) equation
(12) simplifies to
\[
z = r_F - (T - t)^{-1} \ln \left\{ 2N\left(-\frac{1}{2} \sigma \sqrt{T - t}\right) \right\}
\] (13)

It follows that if \( S = B \), the option-adjusted discount rate \( z \) is a function
of the risk-free rate and the time-adjusted volatility. If the pension assets
are invested exactly in the replicating portfolio, then \( S = B \), \( \sigma = 0 \) and
\( z = r_F \). This is intuitive, since in the absence of risk, the market value of
the put option is nil. With respect to the time to maturity \( (T - t) \), note the
following properties
\[
\frac{\partial P_{L_n}}{\partial (T - t)} > 0; \quad \frac{\partial PV(L_n)}{\partial (T - t)} < 0
\] (14)

The first partial derivative shows that the premium for insurance against
a return less than the risk-free rate increases with time. Since the market
value of the pension put increases with time to maturity, the present value
of the benefit decreases with respect to time to maturity. For \( S = B \) the
implied option-adjusted discount rate \( z \) decreases with time to maturity.
This is obvious because the probability of the pension put maturing in the
money (in the case of default) decreases with time. The probability of default
in a risk-neutral world is equal to \( N(-d_2) \) for a put option. However, this
is a limited measure of risk because it does not incorporate the severity of
a default, as discussed in the previous section. Recall that the value of a
put option indeed does take into account the magnitude of the potential
shortfall, see Bodie (1995). He argues that the cost of shortfall insurance
\( P \) is the additional amount of money an investor would have to invest in
a put option to make sure that at maturity his portfolio will have a value
at least as great as it would have been earning the risk-free interest rate.
Thus, for each dollar insured against this shortfall risk, the total amount
invested at the starting date is \( 1 + P \). Define \( X \) as the exercise price of the
put option. From Appendix A.2 it follows that for \( S = X e^{-r_F (T - t)} = 1 \) the
insurance premium per unit of risk is a function of the square root of time
and approximates
\[
\frac{P}{\sigma} \approx \sqrt{(T-t)/2\pi}
\]

(15)

The insurance premium increases with time to maturity and hence also the shortfall risk. The intuition behind this is that as the investment horizon lengthens, the magnitude of the worst possible outcome increases. In the next section, it is highlighted that this property is also valid for mean reverting return processes.

The same line of reasoning as applied above to nominal liabilities can be used for valuing a real pension claim \(L_i\). The present value of a real cash flow at maturity is equal to the market value of a default-free zero-coupon indexed-linked bond \(B_i\) minus the value of the pension put \(P_{L_i}\) with strike price equal to \(L_i\). One additional feature compared to the previous setting, however, is that inflation is a stochastic variable. The exercise price of the pension put is therefore uncertain. Assuming that inflation evolves according to a geometric Brownian motion process the pension put can be modelled as an exchange option, see Margrabe (1978) and Fischer (1978), \((M\text{ refers to Margrabe (1978))}):

\[
PV(L_i) = B_i - P_{L_i}^M
\]

(16)

This is the appropriate valuation model when the ambition in the pension deal is to unconditionally index accrued benefits with the stochastic price inflation, keeping the purchasing power of the deferred wage constant over time. The implied discount rate for a fully funded situation \((S = B_i)\) is again given by equation (12) but with valuation parameter

\[
d_M^i = \left\{ \ln(S/L_i) - \frac{1}{2} \sigma^2(T-t) \right\} / (\sigma \sqrt{T-t})
\]

(17)

Note that the \(\sigma\) in this equation differs from the one in equation (11) because this sigma also takes the correlation between asset returns and inflation into account.

The pension put in this section demonstrates the asymmetric pay-off for the beneficiaries. On the upside, the liabilities are fully paid out at the nominal value of fully indexed with the actual inflation; on the downside, the loss is the total amount of accrued pension rights. This has several disadvantages for the beneficiaries. First, they are overwhelmingly exposed to idiosyncratic risk with respect to deferred income. Second, there is almost perfect correlation between current and deferred income. In addition, they do not gain from the company’s future prospects. In return for paying the
pension put premium, the shareholders receive the unconditional right to an optimal exercise of the pension put. If things turn out negative, the owners of the company will exercise the pension put and write off pension liabilities. In prosperity, however, the owners gain all the upside while the benefits for the pensioners are capped at nominal or fully indexed levels. To fathom the technical details behind this, note that a long position in an option is characterized by convex price movements. The convexity is usually expressed by gamma, which is given by

\[
\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{n(d_1)}{S \sigma \sqrt{T-t}} \\left\{ S \sigma \sqrt{T-t} \right\}
\]

In equation (18) \(n()\) is the density function of the standard normal distribution and \(d_1\) is defined in equation (11). Gamma is positive for long investors: in this case the shareholders of the company. It measures the changes in delta \(\Delta = \frac{\partial P}{\partial S}\) for changes in the underlying asset. A positive gamma implies that delta adjusts favorably to changes in the value of the underlying asset. The delta of a put increases for a falling asset price. The exposure of the writers of the pension put to changes in the market value of the assets backing their pension promises also increases. Beneficiaries in fact are volatility short. This implies that the loss of their position accelerates quickly as the market value of the assets backing their pension promises declines. To summarize this section, if defined pension benefits are subject to the companies default risk the risk-free value must be reduced with the market value of the pension put. This is equivalent to using an option-adjusted discount rate. The pension put indicates that, in absence of a legally separated and capitalized pension fund, the risk for the pension beneficiaries is on the downside. In case of a default of the company the pension rights can be reduced, while on the upside the pension benefits are capped at their nominal or fully indexed level, depending on the promise.

4 General framework for pension fund analysis

The default risk to which the beneficiaries of the pension deal are exposed can be reduced through buying credit risk protection in capital markets or by creating a pension fund. The latter can be seen as a bankruptcy remote special purpose vehicle carrying forward the deferred income until it is payable. To decrease dependence on the sponsor the pension fund should always be fully funded and, e.g. by means of a buffer, derivatives or reinsurance contracts, be able to absorb market and actuarial risks. A clean break between both entities, however, is highly hypothetical.
The general framework for analyzing the relation between sponsor and pension fund is based upon Sharp (1976). Assuming a perfect labor market and rational behavior, employer and employees negotiate labor income $\bar{W}$ equal to the present value of the marginal labor productivity. This reward is subsequently divided into current labor income $W_0$ and the present value of the promised pension benefit $PV(L)$. Sharpe does not specify the nature of $L$ (e.g. nominal or real) but assumes that its value is known for certain.

$$\bar{W} = W_0 + PV(L)$$

(19)

A fund with assets $S$ is set apart from the sponsor to manage the pension obligations. To fully insure the payment of $L$ in the future, the sponsor has to pay a premium in advance for acquiring a put option with equal maturity and exercise price $L$. In addition, the sponsor will claim all assets in excess of $L$ at maturity. This is represented by a short call option with strike price $L$. So, total labor compensation equals

$$\bar{W} = W_0 + S + PL - CL$$

(20)

The put option in equation (20) is different from the pension put in the previous section, where it resembled the default risk of the sponsor. Here it represents a contingent claim of the pension fund on the sponsor which becomes in the money if the pension fund becomes underfunded. The funding decision entails the amount of assets set aside $S$ and the investment policy characterized by $\sigma$, the volatility of the pension fund assets. This funding decision, however, is irrelevant. Changing $S$ or $\sigma$ will be exactly offset by changes in the value of the put and the call. This result is known as the put-call parity: a portfolio consisting of the underlying asset, a long put plus a short call with equal strike prices and time to maturity will always return the risk-free rate. Increasing risk ($\sigma$), for instance, will raise the value of the put and the call by exactly the same amount.

The remainder of this paper analyzes a related pension deal, which basically consists of two distinct promises. This type of pension deal is frequently observed, e.g. in the Netherlands. First, employees are offered a defined benefit related to their final salary or the average pay over their career. This guarantees a minimum nominal pension $L_n$ at retirement $t = T$. This is a single cash flow or bullet. To express the market value of the guaranteed benefit it must be discounted at the risk-free rate $r_F$. This obligation, including actuarial uncertainty, can be fully replicated by investing in a default free zero coupon bond with equivalent maturity. Second, the pension fund aims at providing an indexed pension $L_i$ at maturity ($L_i > L_n$). The ex
ante indexation ambition is denoted by $i$, for instance 2% per annum. The ambition could be linked to the expected inflation or wage growth (both stochastic by nature) over the maturity of pension deal. The actual indexation is conditional on the return on assets over $T - t$ or equivalent to the funding ratio at $t = T$. The funding ratio $F$ expresses the ratio of the market value of the assets $S$ as a percentage of the market value of the nominal liabilities $PV(L_n) = L_n e^{-r_F(T-t)}$. It is assumed that all liabilities are due in year $T$.

5 Unconditional guarantee of the defined benefit

Consider a situation in which the sponsor unconditionally covers all losses in the pension fund. The pension fund in effect has implicitly bought a put option that gives the right to sell the assets to the company at $t = T$ for $L_n$. The pay-off of this put option is $\max(L_n - S_T; 0)$. In return for providing insurance, the company has the unconditional right to withdraw any surpluses in the fund in excess of $L_i$. The pension fund has implicitly written a call option with pay-off $\max(S_T - L_i; 0)$. In absence of counterparty risk, the market value of the pension fund surplus $I$ is given by

$$I = S + P_{L_n} - C_{L_i} - L_n e^{-r_F(T-t)}$$

(21)

Note that the pension fund has no influence on either $L_n$ or $L_i$ because they are given in the pension deal which is negotiated by employers and employees. Following Sharpe (1976) the only parameters to be influenced by the fund are the total amount of assets $S$ and surplus volatility $\sigma$. Figure (3) plots the market value of $I$ as percentage of $L_n$ for different maturities and volatilities. Changing $\sigma$ will have a positive or negative effect on $I$.

Within this setting it is assumed that the pension fund trustees act in the best interest of the participants by maximizing the market value of the indexation contract. Unless stated otherwise it is assumed throughout this paper that the funding ratio ($F$), which is defined as the market value of the assets over the market value of the nominal liabilities, is 100%. So $F = 1$ or $S = L_n e^{-r_F(T-t)}$ and $L_i = L_n e^{i(T-t)}$. From (21), the fund therefore can derive its optimal risk profile by solving

$$\frac{\partial I}{\partial \sigma} = 0 \Rightarrow \frac{\partial P_{L_n}}{\partial \sigma} = \frac{\partial C_{L_i}}{\partial \sigma}$$

(22)

The market value of the pension surplus is maximized when the sensitivities of the market values of both options for changes in surplus volatility
Figure 3: The market value of the pension fund surplus \( (I) \) in equation (21) divided by the market value of \( L_n \) using \( r_F = 0.05, S = 100, i = 0.02, L_n = 100e^{r_F(T-t)} \) and \( L_i = L_ne^{i(T-t)} \).
are equal. The put option provides downside insurance. The call option with the higher exercise price limits the upside potential. In the optimum, the marginal cost of insurance equals the marginal reward for risk taking. The first derivative of the option price with respect to the volatility of the underlying is known as vega. Note that before maturity, there are no intermediate cash outflows between the sponsor and the fund. In the standard Black-Scholes-Merton valuation framework, see Appendix A.3, the vega for a European call or put option on a non-dividend paying asset is given by

\[ Sn(d_1)\sqrt{T-t} \]  

with \( n(d_1) \) the density function of the standard normal distribution or

\[ n(d_1) = e^{-\frac{1}{2}d_1^2}/\sqrt{2\pi} \]  

Thus, equation (22) using (23) simplifies to

\[ n(d_{1,P}) = n(d_{1,C}) \]  

or, applying equation (24)

\[ d_{2,P} = d_{2,C} \]  

The market value of surplus is maximized if the variance of the pension fund surplus is chosen equal to the fixed annual indexation ambition, so

\[ \sigma^* = i \]  

where the asterisk denotes the optimal value. The conversion from equation (26) to (27) is given in Appendix A.4. The interpretation of this result is straightforward. The optimal risk profile solely depends on the indexation ambition. The only uncertainty for the participants in the pension fund is the value of the assets at \( t = T \), within the following boundary conditions. The fund can always sell the assets at \( L_n \) if \( S_T < L_n \) and the sponsor will buy the assets for \( L_i \) if \( S_T > L_i \). In this special case the optimal surplus volatility is independent of \( T - t \). A longer time to maturity does not change the optimal investment policy.

Note that this result is independent of the stochastic process of the underlying assets. Therefore the result is also valid for mean reverting asset return processes. The reason is that option pricing models, such as the Black-Scholes-Merton, are valid regardless of the process for the mean. They are based on the law of one price and the absence of arbitrage profits, see
Bodie (1995) and Lo & Wang (1995). Lo & Wang (1995) provide a simple adjustment for the volatility parameter to correct for autocorrelated asset returns. Although this influences the value of the option, it does not change the (derivation of the) Black-Scholes-Merton formula itself. Note that this paper works the other way around. The pension fund derives its optimal surplus volatility given the indexation ambition and the perceived credit quality of the sponsor. This translates into a particular asset allocation and duration gap given the risk-return characteristics of the available investment opportunities.

The matching cost effective premium for the result in equation (27) is, see Appendix A.4:

$$PV(L) = L_n e^{-r_F(T-t)} \left\{ 2N(-d_2) - N(d_1) + e^{i(T-t)}N(d_2) \right\}$$  \hspace{1cm} (28)

This premium is higher than the premium for a pure nominal benefit but lower than one would have to pay for an unconditional indexed pension, so

$$L_n e^{-r_F(T-t)} < PV(L) < L_i e^{-r_F(T-t)}$$ \hspace{1cm} (29)

Given total labor income $\bar{W}$, current income would equal $W_0 = \bar{W} - PV(L)$. This again shows that, after correction for risk, investing the pension premiums in risky assets does not create value. It is an exchange of current for future income in order to provide inflation protection. As the result in equation (18) is derived assuming continuous time hedging which leads to a risk neutral world, preferences do not play a role in finding the optimal funding strategy.

For $F \neq 1$ or $S \neq L_n e^{-r_F(T-t)}$, the solution to equation (22) is given by

$$\sigma^2 = i - 2(T-t)^{-1} \ln(F)$$ \hspace{1cm} (30)

see also Appendix A.4. This implies that an increasing funding ratio ($F$) lowers the optimal risk profile of the pension fund. This can be explained through the fact that an increasing funding ratio will automatically increase the market value of the refunding option and lowers the market value of the option to increase future premiums. Increasing mismatch risk in that case, is not in the best interest of the beneficiaries of the pension fund. In fact, having a funding ratio in excess of 100% is unnecessary since the downside risk is already fully insured through the put option and does not have to be covered by additional assets in the pension fund.
In the previous section it was already highlighted that one possible drawback in this analysis is the fact that inflation is not known a priori. If the pension fund’s ambition is to compensate for the actual price inflation, $L_i$ should be made stochastic. In that case, the pension fund has written an *exchange call option* on the assets with uncertain strike price $L_i$. The appropriate valuation model in that case is again from Margrabe (1978) and Fischer (1978). The call option in equation (21) becomes an exchange option, i.e. the option to exchange risky assets at an exercise price which is indexed to the uncertain value of the liabilities, see also Blake (1995).

So far we assumed a loyal and solvent sponsor. This leads to an solution in which the optimal risk profile of the pension fund only relates to the indexation ambition expressed in the form of a fixed annual percentage. In reality the behavior of the sponsor will depend on the financial ability to guarantee the accrued pension rights of the retirees. The assumption that the sponsor will bear the full burden of subsequent losses may be too strong. In the next sections this assumption is relaxed in several ways. First, a situation in which the sponsor offers a limited guarantee is considered. Second, the impact on the optimal risk profile in case the sponsor covers a certain percentage of any loss on the pension funds balance sheet is analyzed. Third, the financial ability of the sponsor is included in the model.

### 6 Limited guarantee and partial loss insurance

This section assumes that the sponsor offers a guarantee below a given percentage ($\kappa$) of the accrued benefits $L_n$. In case of default of the pension fund, the beneficiaries lose $(1 - \kappa)L_n$ before the sponsor steps in and covers additional losses. This is a rudimentary way of sharing default risk between the stakeholders. Again, the sponsor successfully claims all assets in excess of $L_i$, causing an asymmetric distribution of investment gains and losses between beneficiaries and the sponsor. The surplus at market value in this set-up is given by

$$ I = S + P_{nL_n} - C_{L_i} - L_n e^{-r_F(T-t)} $$

The annual indexation ambition is again fixed at $i$. The trustees of the pension fund act in the best interest of the beneficiaries by maximizing the market value of $I$ as in equation (22). In Appendix A.5 the optimal surplus volatility is derived as
Figure 4: Values for equation (32) using $r_F = 0.05$, $S = 100$, $i = 0.02$, $L_n = 100e^{r_F(T-t)}$ and $L_i = L_ne^{r(T-t)}$.

$$
\sigma^* = \frac{i^2 - (\ln(1/\kappa)/(T-t))^2}{i + \ln(1/\kappa)/(T-t)}
$$

(32)

For $\kappa = 1$, the sponsor fully guarantees nominal pensions, leading to the result in the section above. Note that for $\kappa < e^{-i(T-t)}$, surplus volatility should equal zero; the fund ought to confine its task to replicating the nominal liabilities in the capital market. If, e.g., $i = 2\%$ and $T - t = 15$ years, the sponsor should at least underwrite 74% of the nominal benefits to make it worthwhile for the pension fund to take on balance sheet risk. In case of a limited guarantee (with boundary conditions $e^{-i(T-t)} < \kappa < 1$), optimal surplus volatility is always less than in the fully assured situation. Also note that duration $(T-t)$ influences the optimal solution. Figure 4 shows the relationship between optimal surplus volatility ($\sigma^*$), the fraction of defined benefits underwritten by the sponsor ($\kappa$) and time to maturity $(T-t)$. A lower $\kappa$ is partially offset by the time to maturity: a pension scheme with a longer duration can engage somewhat more risk.

The next step is to consider a situation in which shortages in the pension fund are partially shared between the beneficiaries and the sponsor. The pay-off of the put at maturity equals $\lambda \max(L_n - S_T; 0)$ in case of default, with $0 \leq \lambda \leq 1$. Factor $1 - \lambda$ resembles a depreciation factor of the defined benefits for the beneficiaries in case of unforeseen cumulated investment.
Figure 5: Values for equation (34) using $r_F = 0.05$, $S = 100$, $i = 0.02$, $L_n = 100e^{r_F T}$ and $L_i = L_n e^{i(T-t)}$.

losses at maturity. The sponsor finances the remainder of the loss. Surplus at current market value is

$$I = S + \lambda P_{L_n} - C_{L_i} - L_n e^{-r_F(T-t)}$$

Solving $\partial I / \partial \sigma = 0$ gives the following relationship between the indexation target, time to maturity and $\lambda$

$$\sigma^* = \sqrt{\frac{i^2}{i - 2 \ln(\lambda)/(T - t)}}$$

If the counterparty of the insurance contract covers all losses ($\lambda = 1$), again equation (27) results. For $\lambda \to 0$ preferred surplus volatility also converges to zero. Figure 5 shows the relationship between $\sigma^*$, $\lambda$ and $T - t$. The above differs from the example in the previous paragraph in that any positive surplus volatility is always optimal. This is based upon the assumption that losses are divided between the sponsor and the beneficiaries. In the case of a limited guarantee there is a boundary condition because the sponsor bears all the risk below threshold $\kappa$. 
7 Sponsor vulnerability

The preceding section assumed that loss absorption occurs according to fixed parameters and is known in advance. Typically, pension schemes rarely include such explicit arrangements. The quality of the sponsor guarantee generally will depend on the sponsor’s financial ability to underwrite losses. Extending the analysis further, specific characteristics of the sponsor are brought into the equation. The main feature is that the put option in equation (21) is a so-termed vulnerable option: a derivative security with the risk of a defaulting counterparty. It is straightforward that options which are vulnerable to counterparty credit risk have lower market values than otherwise identical but non-vulnerable options. These options are described in Johnson & Stulz (1987), Hull & White (1995) and Klein (1996). For an overview see Ammann (2001).

This paper applies the closed form formula from Klein (1996). Let $V$ be the current market value of the sponsor (time subscript $t$ is suppressed) and $D_T$ the future total (fixed) liabilities of the sponsor including those potentially arising from underfunding at the pension fund level. All liabilities have the same maturity. Furthermore, Klein (1996) distinguishes deadweight losses associated with bankruptcy expressed as a percentage of the market value of the assets of the counterparty ($\alpha$). These losses include the direct cost of the bankruptcy, reorganization expenses and the effects of distress on the business operations of the company. Key in this set-up is that at $t = T$ default of the company is triggered if $V_T < D_T$. The market value of the pension fund surplus is equal to

$$ I = S + P^v_{L_n} - C_{L_i} - L_n e^{-r_F(T-t)} $$

with the market value of the vulnerable put option $P^v_{L_n}$ equal to

$$ P^v_{L_n} = L_n e^{-r_F(T-t)} N_2(-b_1, b_2, \rho) - S N_2(-a_1, a_2, \rho) $$
$$ + (1 - \alpha) \frac{V}{D_T} \left\{ L_n N_2(-d_1, d_2, -\rho) - S e^{(r_F + \rho \sigma_F)(T-t)} N_2(-c_1, c_2, -\rho) \right\} $$

The first two terms of this equation are basically similar to a regular, default-free, put option. The last term relates to the bankruptcy costs, to the current sponsor’s financial position $V/D_T$ and the interdependence between the sponsor and its pension fund. Symbol $\rho$ represents the correlation between the sponsor’s equity and the pension fund’s surplus. The volatility
of the sponsor’s equity is $\sigma_V$ and $N_2()$ is the cumulative bivariate normal density function. The remaining variables are explained in Appendix A.6. The optimal surplus volatility can only be found by numerical procedures. For the special case of zero correlation between the fund and the sponsor ($\rho = 0$), the optimal risk profile reduces to the following expression, which is derived in Appendix A.6

$$\sigma^* = \frac{i^2}{i - 2\ln \left\{ N(a_2) + (1 - \alpha) \frac{V}{DT} e^{r_F(T-t)} N(c_2) \right\} / (T-t)}$$

(37)

Note that for $V >> DT$ there is virtually no default risk for the fund’s beneficiaries. In that case, the optimum again equals $\sigma^* = i$ which is also the upper limit of the feasible risk profiles. Table (1) shows how surplus volatility is conditional on distinct characteristics of the sponsor.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.50$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_V = 0.10$</td>
<td>0.139</td>
<td>0.135</td>
<td>0.131</td>
<td>0.128</td>
<td>0.125</td>
</tr>
<tr>
<td>$\sigma_V = 0.25$</td>
<td>0.095</td>
<td>0.087</td>
<td>0.080</td>
<td>0.074</td>
<td>0.069</td>
</tr>
<tr>
<td>$\sigma_V = 0.50$</td>
<td>0.057</td>
<td>0.054</td>
<td>0.051</td>
<td>0.049</td>
<td>0.046</td>
</tr>
<tr>
<td>$\rho = 0$</td>
<td>0.096</td>
<td>0.090</td>
<td>0.086</td>
<td>0.082</td>
<td>0.079</td>
</tr>
<tr>
<td>$\rho = 0.25$</td>
<td>0.081</td>
<td>0.075</td>
<td>0.070</td>
<td>0.066</td>
<td>0.062</td>
</tr>
<tr>
<td>$\rho = 0.50$</td>
<td>0.067</td>
<td>0.065</td>
<td>0.063</td>
<td>0.061</td>
<td>0.059</td>
</tr>
<tr>
<td>$V/DT = 2.0$</td>
<td>0.111</td>
<td>0.102</td>
<td>0.095</td>
<td>0.089</td>
<td>0.084</td>
</tr>
<tr>
<td>$V/DT = 0.5$</td>
<td>0.071</td>
<td>0.064</td>
<td>0.059</td>
<td>0.054</td>
<td>0.049</td>
</tr>
<tr>
<td>$V/DT = 0.2$</td>
<td>0.051</td>
<td>0.046</td>
<td>0.042</td>
<td>0.039</td>
<td>0.034</td>
</tr>
</tbody>
</table>

†Using vulnerable option valuation formula from Klein (1996) with defaults $S = 100$, $r_F = 0.05$, $i = 0.02$, $T - t = 15$, $L_n = S e^{r_F(T-t)}$, $L_i = L_n e^{i(T-t)}$ and various values of $\alpha$, $\sigma_V$, $\rho$ and $V/DT$.

As one would expect, the table shows that there is a relationship between the volatility of the sponsor $\sigma_V$ and optimal surplus volatility at the pension fund level $\sigma^*$. If the sponsor has a high risk profile, the associated pension fund should reduce risk taking. This is also the case for the correlation between the sponsor and the fund. As already mentioned before, an increasing ratio of the market value of the sponsor to the notional value of all debt ($V/DT$), provides the pension fund with additional risk taking resources. The quality of the sponsor guarantee increases for the beneficiaries.
because the sponsor has less outstanding debt relative to its own market value. The impact of bankruptcy costs \( \alpha \) on \( \sigma^* \) is limited. Although ranging from \( \alpha = 0 \) to \( \alpha = 1 \), the average reduction in surplus volatility may count for a few percentage points.

8 Portfolio weights

This section explains how the results from the previous sections translate into optimal portfolio weights. For that the surplus volatility must be related to the underlying variables. Consider two possible portfolios: a fully diversified equity portfolio and a bond index portfolio with a given duration. The fraction invested in stocks is \( w \). The market value of the surplus at \( t+1 \) equals accrued market value of total assets minus accrued market value of the nominal liabilities

\[
I_{t+1} = S \{ w(1 + r_E) + (1 - w)(1 + r_B) \} - PV(L_n)(1 + r_L) \tag{38}
\]

where \( r_E \) reflects the expected return on equities and \( r_B \) the expected return on bonds. The return on the nominal liabilities \( r_{L_n} \) can be replicated from a leveraged position in the bond index portfolio. The return on liabilities equals

\[
r_{L_n} = r_F + \beta (r_B - r_F) \tag{39}
\]

where \( \beta \) is a leverage factor which reflects the duration of the pension liabilities relative to the duration of the investable bond index. Hence, the market value of the surplus one period ahead can also be written as

\[
I_{t+1} = S \{ w(1 + r_E) + (1 - w)(1 + r_B) \} - PV(L_n) \{ 1 + r_F + \beta(r_B - r_F) \} \tag{40}
\]

Equivalently, the surplus expressed as a percentage of the market value of total assets equals

\[
\frac{I_{t+1}}{S} = w(1+r_E) + \left( 1 - w - \frac{PV(L_n)}{S} \right) r_B + 1 - w - \frac{PV(L_n)}{S} (1 + r_F - \beta r_F) \tag{41}
\]

Since the variance of \( r_F \) and the other constants are zero by definition, the variance of this surplus ratio is
Var \left( \frac{I_{t+1}}{S} \right) = w^2 \sigma_E^2 + (1 - w - \beta/F)^2 \sigma_B^2 + 2w(1 - w - \beta/F)\rho\sigma_E\sigma_B \quad (42)

where \( \rho \) represents the correlation between equity and bond returns and \( F = S/PV(L_n) \) is the funding ratio. If the duration of the bond portfolio is equal to the duration of the pension liabilities \( (\beta = 1) \) and the current funding ratio is 100% \( (F = 1) \), the optimal percentage of risky assets in the investment portfolio equals (see Appendix A.7)

\[
w = \frac{\sigma^*}{\sqrt{\sigma_E^2 + \sigma_B^2 - 2\rho\sigma_E\sigma_B}} \quad (43)
\]

where \( \sigma^* \) is given in equations (27), (32), (34) or (37). Again the weights are independent of expected returns. Remember that this comes from the fact that the optimum is derived under the assumption of continuous time hedging which leads to a risk-neutral world. Table (2) gives some indicative numbers for the optimal weight of risky assets in the portfolio. The leverage factor \( \beta \), or duration gap between assets and liabilities, strongly influences the amount available for investments in risky assets. The intuition behind this is that a large duration gap between the pension liabilities and the bond portfolio absorbs a large part of the available risk budget. The impact of the correlation between equities and bonds has less impact. A positive correlation means that equity returns behave more like bond returns, which would imply that they are a better hedge for (nominal) defined benefit liabilities. This explains why the optimal weight in risky assets decreases for negative correlations.

<table>
<thead>
<tr>
<th>Leverage</th>
<th>( \beta = 0.5 )</th>
<th>( \beta = 1.0 )</th>
<th>( \beta = 1.5 )</th>
<th>( \beta = 2.0 )</th>
<th>( \beta = 2.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = -0.25 )</td>
<td>0.741</td>
<td>0.624</td>
<td>0.459</td>
<td>0.241</td>
<td>-0.062</td>
</tr>
<tr>
<td>( \rho = 0.0 )</td>
<td>0.771</td>
<td>0.687</td>
<td>0.535</td>
<td>0.304</td>
<td>-0.097</td>
</tr>
<tr>
<td>( \rho = 0.25 )</td>
<td>0.811</td>
<td>0.774</td>
<td>0.646</td>
<td>0.407</td>
<td>-0.247</td>
</tr>
</tbody>
</table>

\( \dagger \) Using \( F = 1.0 \), \( \sigma_E = 0.18 \) and \( \sigma_B = 0.10 \).

9 Possible extensions

The results in this paper provide tools for evaluating the quality of defined benefit pension savings with conditional indexation. However, the analytical solutions depend on several strong assumptions. Possible extensions of the
analysis can be derived from weakening these assumptions. For instance, it is assumed that there is only one objective, namely that the trustees of the pension fund act in the best interest of the beneficiaries by maximizing the market value of the pension fund surplus. This can be extended to a specification in which both the interests of the beneficiaries as well as the sponsor are included. In addition the paper assumes market completeness and rational behavior, which relieves the necessity of specifying utility functions. Also the possibility of virtually cost-less continuous time hedging may be relaxed, although large pension funds are known to frequently rebalance their portfolios in a very cost-effective manner. The introduction of intermediate cash flows between sponsor and pension fund is a useful extension as well as multi-period pension liabilities and explicit modelling of actuarial risks. Finally, including a stochastic term structure of interest rates and stochastic inflation will lead to even more realistic, but probably less analytical, results.

10 Summary

This paper analyzes the market value based balance sheet of a pension fund with conditionally indexed defined benefit liabilities. The market value of the indexation clause can be derived from the difference between an implicit put and call option on the assets of the pension fund. The put resembles future premium increases to the pension fund, the call future refunding to the sponsor. When these derivatives are analyzed as being traded on regulated markets, the optimal risk profile depends only on the indexation ambition, expressed in the form of a fixed annual target rate. Traded derivatives are virtually free from counterparty credit risk through the clearing and settlement function of the exchange. In the context of a pension fund, sponsor vulnerability reduces the quality of the downside insurance for the beneficiaries of a defined benefit pension scheme. Unlike on regulated markets, between a pension fund and its sponsor there are no margin requirements and, in many cases, not even explicit financing arrangements. This gives the sponsor the upper hand in the game of sharing the residual risk at the pension fund level. Residual risk is a loss that cannot be absorbed and ultimately leads to a write-off of accrued benefits. The beneficiaries of the pension fund in fact are confronted with counterparty credit risk. This is shown by correcting the market value of the option to increase future premiums for the financial ability of the sponsor to actually do so. This paper suggests that given a situation in which the sponsor unconditionally claims
surplus assets but is reluctant or unable to fully cover losses, there is an asymmetric allocation of the residual risk over the sponsor and the participants in the fund. In such a situation and under the assumption that the pension fund maximizes the market value of its surplus, it is optimal for the fund to reduce risk-taking, which means that the fund cannot fully pursue its indexation policy.
References


A Technical Appendices

A.1 Pension fund risk management

This appendix covers some basic topics in long-term risk management. In this section we assume that the deterministic nominal liability of the pension fund at $t = T$ is $L_n$. Central assumption is that asset prices follow a geometric Brownian motion according to

$$dS = \mu S dt + \sigma S d\omega$$

with constants $\mu$ the expected return per unit time and $\sigma^2$ the variance of returns per unit time. Source of uncertainty is a Wiener process $\omega$. Under this assumption the change in the logarithm of price is Gaussian distributed. If the asset price $S$ is described by a geometric Brownian motion, any function of $S$ is the solution of the partial differential equation obtained from Itô’s lemma, Hull (2003)

$$dG = \left[ \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right] dt + \frac{\partial G}{\partial S} \sigma S d\omega$$

The stochastic process for $G = \ln(S)$ therefore can be derived from

$$dG = \left[ \frac{1}{S} \mu S + 0 - \frac{1}{2} \frac{1}{S^2} \sigma^2 S^2 \right] dt + \frac{1}{S} \sigma S d\omega$$
or

$$dG = \left[ \mu - \frac{1}{2} \sigma^2 \right] dt + \sigma d\omega$$

If $S$ is lognormal distributed, the change in $G = \ln(S)$, or the continuously compounded return, over $t = 0$ to $t = T$ is characterized by a normal distribution

$$\ln(S_T/S) \sim \Phi \left( \left( \mu - \frac{1}{2} \sigma^2 \right) (T-t), \sigma \sqrt{T-t} \right)$$

where $\Phi$ represents the normal distribution. Define standard normal distributed variable $\tilde{z} \sim N(0, 1)$ as

$$\tilde{z} = \frac{\ln(S_T/S) - m(T-t)}{\sigma \sqrt{T-t}}$$

with $m = \mu - \frac{1}{2} \sigma^2$. This means that the terminal asset value of the pension fund can be written as
\( S_T = Se^{m(T-t)+\sigma \sqrt{T-t}z} \)

The expect value

\[
E(S_T) = SE \left( e^{m(T-t)+\sigma \sqrt{T-t}z} \right) \\
= Se^{m(T-t)} \int_{-\infty}^{\infty} e^{\sigma \sqrt{T-t}z} e^{-\frac{1}{2}z^2} \frac{1}{\sqrt{2\pi}} dz \\
= S \left( e^{m(T-t)} + e^{\frac{1}{2}\sigma^2(T-t)} \right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma \sqrt{T-t})^2} \frac{1}{\sqrt{2\pi}} dz = Se^{m(T-t)+\frac{1}{2}\sigma^2(T-t)} = Se^{\mu(T-t)} 
\]

The expected terminal market value of the pension fund assets therefore equals

\[ E(S_T) = Se^{\mu(T-t)} \]

The pension fund defaults if the market value of the assets at maturity is less than the guaranteed pension benefits \( L_n \). The probability of default can be derived from, with \( f(S_T) \) the log normal density function for \( S_T \)

\[
PD = P(S_T < L_n) = \int_0^{L_n} f(S_T)dS_T \\
= \int_{-\infty}^{\ln(L_n/S)-m(T-t)} e^{-\frac{1}{2}z^2} \frac{1}{\sqrt{2\pi}} dz \\
= N \left( \frac{\ln(L_n/S) - m(T-t)}{\sigma \sqrt{T-t}} \right) 
\]

Using the following property of the normal distribution \( N(x) = 1 - N(-x) \) results in

\[
P(S_T < L_n) = 1 - N \left( \frac{\ln(S/L_n) + m(T-t)}{\sigma \sqrt{T-t}} \right) 
\]

Define (using bold face to distinguish from the equivalent risk-neutral parameter from the Black-Scholes-Merton framework)
\[ d_2 = \frac{\ln(S/L_n) + m(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln(S/L_n) + (\mu - \frac{1}{2}\sigma^2) (T-t)}{\sigma \sqrt{T-t}} \]

thus

\[ PD = P(S_T < L_n) = 1 - N(d_2) = N(-d_2) \]

This expression can be used to analyze the relation between the probability of underfunding \( N(-d_2) \), time to maturity \((T-t)\) and the funding ratio \((F)\). First use \( F = Se^{-r_F(T-t)}/L_n \) to rewrite \( d_2 \)

\[ d_2 = \frac{\ln(F) + (\mu - r_F - \frac{1}{2}\sigma^2) (T-t)}{\sigma \sqrt{T-t}} \]

To determine for which time to maturity the probability of underfunding is at maximum value, derive

\[ \frac{\partial d_2}{\partial T-t} = \frac{(\mu - r_F - \frac{1}{2}\sigma^2) \sigma \sqrt{T-t} - \frac{1}{2}\sigma (T-t)^{-\frac{1}{2}} (\ln(F) + (\mu - r_F - \frac{1}{2}\sigma^2) (T-t))}{\sigma^2(T-t)} = 0 \]

\[ (\mu - r_F - \frac{1}{2}\sigma^2) \sqrt{T-t} - \frac{1}{2}(T-t)^{-\frac{1}{2}} \ln(F) - \frac{1}{2} (\mu - r_F - \frac{1}{2}\sigma^2) \sqrt{T-t} = 0 \]

This results in

\[ T - t^* = \ln(F)/\left(\mu - r_F - \frac{1}{2}\sigma^2\right) \]

The loss given default can be derived from evaluating the conditional terminal asset value

\[ E(S_T|S_T < L_n) = \frac{\int_0^{L_n} S_T f(S_T) dS_T}{P(S_T < L_n)} \]

\[ = \int_{-\infty}^{\frac{\ln(L_n/S) - m(T-t)}{\sigma \sqrt{T-t}}} S e^{m(T-t)+\sigma \sqrt{T-t}z} e^{-\frac{1}{2}z^2} \frac{1}{\sqrt{2\pi}} dz/N(-d_2) \]
To simplify define $y = z - \sigma \sqrt{T - t}$ with $\partial y / \partial z = 1$, so that

$$E(S_T|S_T < L_n) = S e^{m(T-t)} + \frac{1}{2} \sigma^2(T-t) \int_{-\infty}^{\frac{\ln(L_n/S) - m(T-t)}{\sigma \sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} \ e^{-\frac{1}{2} y^2} dy / N(-d_2)$$

Substituting $m = \mu - \frac{1}{2} \sigma^2$ gives

$$= S e^{m(T-t)} + \frac{1}{2} \sigma^2(T-t) N \left( \frac{\ln(L_n/S) - m(T-t)}{\sigma \sqrt{T-t}} - \sigma \sqrt{T-t} \right) / N(-d_2)$$

which leads to the result that

$$LGD = E(S_T|S_T < L_n) = S e^{\mu(T-t)} N (-d_1) / N(-d_2)$$

with

$$d_1 = \frac{\ln(S/L_n) + (\mu + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

**A.2 Shortfall insurance**

Bodie (1995) argues that the cost of shortfall insurance $P$ is the additional amount of money an investor has to add to his risky investment to ensure that at maturity the portfolio will have a value at least as great as it would have earning the risk-free interest rate. Thus, for each dollar insured against this particular shortfall, the total amount invested at the starting date is $1 + P$. The Black-Scholes-Merton pricing formula for this put option is
\[ P = X e^{-r(T-t)} N(-d_2) - SN(-d_1) \]

For
\[ S = X e^{-r(T-t)} = 1 \]

which implies that
\[ d_1 = \frac{1}{2} \sigma \sqrt{T-t}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \]

the value of the put reduces to
\[ P = N(d_1) - N(d_2) \]

where \( N(d) \) represents the cumulative normal density function
\[ N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2} z^2} \, dz = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{d}{\sqrt{2}} \right) \right] \]

The \( \text{erf} \) function can be described using a Maclaurin series
\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \]

From this the first order approximation, which is adequate only for \( x \ll 1 \), is
\[ \text{erf}(x) \approx \frac{2x}{\sqrt{\pi}} \]

From this the insurance premium is a function of the volatility and the time to maturity
\[ P \approx \sigma \sqrt{(T-t)/2\pi} \]

**A.3 Vega**

The partial derivative of the price of a call or put option with respect to volatility is known as vega. The closed form solution to \( \partial C / \partial \sigma \) is can be found by analyzing the relation between valuation parameters \( d_2 \) and \( d_1 \) which is given by
\[ d_2 = d_1 - \sigma \sqrt{T-t} \]
Taking the square of both sides of this equation leads to

\[ d_2^2 = d_1^2 - 2d_1 \sigma \sqrt{T-t} + \sigma^2 (T-t) \]

The definition of \( d_1 \) is

\[ d_1 = \frac{\ln(S/X) + \left( r_F + \frac{1}{2} \sigma^2 \right) (T-t)}{\sigma \sqrt{T-t}} \]

where \( X \) is the strike price of the option. Substituting this definition in the previous relation results in

\[ d_2^2 = d_1^2 - 2 \left\{ \ln(S/X) + \left( r_F + \frac{1}{2} \sigma^2 \right) (T-t) \right\} + \sigma^2 (T-t) \]

which can be simplified to

\[ d_2^2 = d_1^2 - 2 \ln(S e^{r_F(T-t)} / X) \]

This expression for \( d_2^2 \) can be used to quantify the density function of the standard normal distribution

\[ n(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_2^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_2^2 + \ln(S e^{r_F(T-t)} / X)} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_2^2} S e^{r_F(T-t)} / X = n(d_1) S e^{r_F(T-t)} / X \]

Using this relation, the vega of a European call option on a non-dividend paying asset follows from differentiation of the call price with respect to volatility

\[ \text{vega}_C = \frac{\partial C}{\partial \sigma} = S \frac{\partial N(d_1)}{\partial \sigma} - X e^{-r_F(T-t)} \frac{\partial N(d_2)}{\partial \sigma} \]

\[ = Sn(d_1) \frac{\partial d_1}{\partial \sigma} - X e^{-r_F(T-t)} n(d_2) \frac{\partial d_2}{\partial \sigma} \]

\[ = Sn(d_1) \frac{\partial d_1}{\partial \sigma} - X e^{-r_F(T-t)} \left\{ n(d_1) S e^{r_F(T-t)} / X \right\} \frac{\partial d_2}{\partial \sigma} \]

\[ = Sn(d_1) \left\{ \frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} \right\} \]

with
\[
\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \left\{ -\frac{\ln(S/X)}{\sigma^2 \sqrt{T-t}} + \frac{1}{2} \sqrt{T-t} \right\} - \left\{ -\frac{\ln(S/X)}{\sigma^2 \sqrt{T-t}} - \frac{1}{2} \sqrt{T-t} \right\} = \sqrt{T-t}
\]

Hence

\[\text{vega}_C = Sn(d_1) \sqrt{T-t}\]

This is also the vega of an equivalent put option.

### A.4 Unconditional guarantee

The pension fund aims at maximizing the market value of the indexation contract. From equation (21) the fund therefore can derive its optimal risk profile by solving

\[
\frac{\partial I}{\partial \sigma} = 0 = > \frac{\partial P_{\text{Lm}}}{\partial \sigma} = \frac{\partial C_{\text{Li}}}{\partial \sigma}
\]

The market value of the pension surplus is maximized when the sensitivities of the market values of both options for changes in surplus volatility are equal. I.e. the vega of the put equals the vega of the call, so

\[Sn(d_{1,P}) \sqrt{T-t} = Sn(d_{1,C}) \sqrt{T-t}\]

or

\[n(d_{1,P}) = n(d_{1,C})\]

Using the definition of density function of standardized normal variable

\[n(d_1) = e^{-\frac{1}{2}d_1^2}/\sqrt{2\pi}\]

gives

\[d_{1,P}^2 = d_{1,C}^2\]

Assuming a funding ratio of 100\% (\(F = 1\)) or \(S = L_n e^{-r F (T-t)}\), so that \(L_n = S e^{r F (T-t)}\) and \(L_i = L_n e^{i (T-t)}\), the option valuation parameters are defined by

\[d_{1,P} = \frac{\ln \left( \frac{S}{Se^{r F (T-t)}} \right) + (r_F + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} = \frac{1}{2} \sigma^2 (T-t) \]

\[\text{vega}_C = Sn(d_1) \sqrt{T-t}\]
\[ d_{1,C} = \frac{\ln \left( \frac{S}{e^{r_F(T-t)}} \right) + (r_F + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} = \frac{-i(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

Therefore \( d_{2,1,P} = d_{2,1,C} \) equals

\[ \left\{ \frac{1}{2}\sigma^2(T-t) \right\}^2 = \left\{ -i(T-t) + \frac{1}{2}\sigma^2(T-t) \right\}^2 \]

This reduces simply to

\[ \sigma^* = i. \]

The corresponding premium can be derived from

\[ PV(L) = PV(L_0) + P_L - C_L \]

or equivalently

\[ PV(L) = L_n e^{-r_F(T-t)} \left\{ 2N(-d_{2,P}) - N(d_{1,C}) + e^{i(T-t)}N(d_{2,C}) \right\} \]

For \( F \neq 1 \) the option valuation parameters are given by

\[ d_{1,P} = \frac{\ln(F) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

and

\[ d_{1,C} = \frac{\ln(F) - i(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

Solving \( d_{2,1,P} = d_{2,1,C} \) in this case results in the following expression for the optimum

\[ \sigma^* = i - \frac{2\ln(F)}{T-t} \]

### A.5 Limited guarantee and partial loss insurance

In the case of a limited guarantee the exercise price of the put option \( L_n \) is multiplied by factor \( \kappa \) to get

\[ \frac{\partial I}{\partial \sigma} = 0 \Rightarrow \frac{\partial P_{\kappa L_n}}{\partial \sigma} = \frac{\partial C_{Li}}{\partial \sigma} \]
Where parameter $d_1$ is adjusted accordingly

$$d_{1,P} = \frac{\ln \left( \frac{S_{\text{S}e^{r_{\text{F}}(T-t)}}}{\kappa S_{\text{e}^{r_{\text{F}}(T-t)}}} \right) + (r_{\text{F}} + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln(1/\kappa) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}}$$

With this adjustment $d_{1,P}^2 = d_{1,C}^2$ results in

$$\left\{ \frac{\ln(1/\kappa) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \right\}^2 = \left\{ \frac{-i(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \right\}^2$$

The risk profile maximizing the market value of surplus is given by

$$\sigma^2 = \frac{i^2 - (\ln(1/\kappa)/(T-t))^2}{i + \ln(1/\kappa)/(T-t)}$$

In the case of partial loss insurance the problem is as follows

$$\frac{\partial I}{\partial \sigma} = 0 \Rightarrow \lambda \frac{\partial P_{\text{Ln}}}{\partial \sigma} = \frac{\partial C_{Li}}{\partial \sigma}$$

This can be expressed as

$$\lambda \exp(-\frac{1}{2}d_{1,P}^2) = \exp(-\frac{1}{2}d_{1,C}^2)$$

Taking the log of both sides leads to

$$d_{1,P}^2 - 2\ln \lambda = d_{1,C}^2$$

$$\left\{ \frac{\frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \right\}^2 - 2\ln \lambda = \left\{ \frac{-i(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \right\}^2$$

This equation can be simplified to

$$\sigma^2 = \frac{i^2}{i - 2\ln(\lambda)/(T-t)}$$
A.6 Vulnerable option

For the market value of vulnerable put option we apply Klein (1996).

\[
P_{Ln}^v = L_n e^{-r_F(T-t)} N_2(-b_1, b_2, \rho) - SN_2(-a_1, a_2, \rho) + (1 - \alpha) \frac{V}{DT} \left\{ L_n N_2(-d_1, d_2, -\rho) - S e^{(r_F+\rho \sigma_V)(T-t)} N_2(-c_1, c_2, -\rho) \right\}
\]

With \( N_2() \) cumulative bivariate normal density function. The valuation parameters and partial derivatives are

\[
a_1 = \ln \left( \frac{S}{S_{e^{r_F(T-t)}}} \right) + (r_F + \frac{1}{2} \sigma^2)(T - t) \quad \frac{\partial a_1}{\partial \sigma} = \frac{1}{2} \sqrt{T - t}
\]

\[
a_2 = \ln \left( \frac{V}{DT} \right) + (r_F - \frac{1}{2} \sigma_V^2 + \rho \sigma_V)(T - t) \quad \frac{\partial a_2}{\partial \sigma} = \rho \sqrt{T - t}
\]

\[
b_1 = \ln \left( \frac{S}{S_{e^{r_F(T-t)}}} \right) + (r_F - \frac{1}{2} \sigma^2)(T - t) \quad \frac{\partial b_1}{\partial \sigma} = -\frac{1}{2} \sqrt{T - t}
\]

\[
b_2 = \ln \left( \frac{V}{DT} \right) + (r_F - \frac{1}{2} \sigma_V^2)(T - t) \quad \frac{\partial b_2}{\partial \sigma} = 0
\]

\[
c_1 = \ln \left( \frac{S}{S_{e^{r_F(T-t)}}} \right) + (r_F + \frac{1}{2} \sigma^2 + \rho \sigma_V)(T - t) \quad \frac{\partial c_1}{\partial \sigma} = \frac{1}{2} \sqrt{T - t}
\]

\[
c_2 = -\ln \left( \frac{V}{DT} \right) + (r_F + \frac{1}{2} \sigma_V^2 + \rho \sigma_V)(T - t) \quad \frac{\partial c_2}{\partial \sigma} = -\rho \sqrt{T - t}
\]

\[
d_1 = \ln \left( \frac{S}{S_{e^{r_F(T-t)}}} \right) + (r_F - \frac{1}{2} \sigma^2 + \rho \sigma_V)(T - t) \quad \frac{\partial d_1}{\partial \sigma} = \frac{1}{2} \sqrt{T - t}
\]
\[ d_2 = -\frac{\ln \left( \frac{V}{D_T} \right) + (r_F + \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}} \quad \frac{\partial d_2}{\partial \sigma} = 0 \]

The partial derivative of the value of the put with respect to surplus volatility is

\[
\frac{\partial P^v}{\partial \sigma} = Sn(-b_1)N \left( \frac{b_2 + b_1 \rho}{\sqrt{1 - \rho^2}} \right) \frac{\partial - b_1}{\partial \sigma} \nabla - S \left\{ n(-a_1)N \left( \frac{a_2 + a_1 \rho}{\sqrt{1 - \rho^2}} \right) \frac{\partial - a_1}{\partial \sigma} + n(a_2)N \left( \frac{-a_1 - a_2 \rho}{\sqrt{1 - \rho^2}} \right) \frac{\partial a_2}{\partial \sigma} \right\} 
\]

\[-\rho \sigma \sigma \nu(T-t)(1-\alpha) \frac{V}{D_T} e^{r_F(T-t)} e^{\rho \sigma \sigma \nu(T-t)} N_2(-c_1, c_2, -\rho) \]

\[-(1-\alpha) \frac{V}{D_T} e^{r_F(T-t)} e^{\rho \sigma \sigma \nu(T-t)} \left\{ n(-c_1)N \left( \frac{c_2 - c_1 \rho}{\sqrt{1 - \rho^2}} \right) \frac{\partial - c_1}{\partial \sigma} + n(c_2)N \left( \frac{-c_1 + c_2 \rho}{\sqrt{1 - \rho^2}} \right) \frac{\partial c_2}{\partial \sigma} \right\} \]

Solving for \( \rho = 0 \) gives the following reduced formula for the sensitivity with respect to volatility

\[
\frac{\partial P^v}{\partial \sigma} = Sn(a_1) \sqrt{T-t} \left\{ N(a_2) + (1-\alpha) \frac{V}{D_T} e^{r_F(T-t)} N(c_2) \right\} 
\]

Deriving \( \frac{\partial P^v_{\ln}}{\partial \sigma} = \frac{\partial C_{Li}}{\partial \sigma} \) results in

\[ n(a_1) \left\{ N(a_2) + (1-\alpha) \frac{V}{D_T} e^{r_F(T-t)} N(c_2) \right\} = n(d_1) \]

which can be written as

\[
\frac{-i^2(T-t) + i \sigma^2(T-t)}{\sigma^2} = 2 \ln \left\{ N(a_2) + (1-\alpha) \frac{V}{D_T} e^{r_F(T-t)} N(c_2) \right\} 
\]
Given zero correlation between the pension fund and the sponsor the optimal surplus variance equals

$$\sigma^* = \frac{i^2}{i - 2 \ln \left\{ N(a2) + (1 - \alpha) \frac{V}{D^p} e^{\gamma(T-t)} N(c2) \right\} / (T-t)}$$

### A.7 Portfolio weights

From equation (42) if follows that the portfolio weights must be chosen so that surplus variance equals $\sigma^2$

$$w^2 \sigma_E^2 + (1 - w - \beta F)^2 \sigma_B^2 + 2w(1 - w - \beta F)\rho \sigma_E \sigma_B = \sigma^2$$

where $w$ represents the percentage to invest in a fully diversified equity portfolio. Simplifying results in

$$w^2 \sigma_E^2 + (1 + w^2 + \beta^2 F^2 - 2w - 2\beta F + 2w\beta F) \sigma_B^2 + (2w^2 - 2w - 2w\beta F) \rho \sigma_E \sigma_B = \sigma^2$$

Which can be rearranged into

$$w^2 \left\{ \sigma_E^2 + \sigma_B^2 - 2\rho \sigma_E \sigma_B \right\} + w \left\{ -2\sigma_B^2 + 2\beta F \sigma_B^2 + 2\rho \sigma_E \sigma_B - 2\beta F \rho \sigma_E \sigma_B \right\} + (1 + \beta^2 F^2 - 2\beta F) \sigma_B^2 = \sigma^2$$

Define

$$A = \sigma_E^2 + \sigma_B^2 - 2\rho \sigma_E \sigma_B$$

$$B = -2\sigma_B^2 + 2\beta F \sigma_B^2 + 2\rho \sigma_E \sigma_B - 2\beta F \rho \sigma_E \sigma_B = 2\sigma_B^2 (\beta F - 1) - 2\rho \sigma_E \sigma_B (\beta F - 1)$$

$$C = (\beta^2 F^2 - 2\beta F + 1)\sigma_B^2 - \sigma^2$$

So that $w^*$ can be derived from

$$w^2 A + wB + C = 0$$

which has two possible solutions. However since the expected return on equities exceeds the expected return on bonds ($r_E > r_B$) the solution which gives the highest $w$ is preferred, so that the optimal weight is
\[ w^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \]

A special case arises when the duration of the bond portfolio equals the duration of the liabilities and the funding ratio equals 100%, specifically if \( \beta = F = 1 \rightarrow B = 0 \land C = -\sigma^2 \), which results in

\[ w^* = \frac{\sqrt{4A\sigma^2}}{2A} = \frac{\sigma^*}{\sqrt{\sigma_L^2 + \sigma_B^2 - 2\rho \sigma_E \sigma_B}} \]

This is expression (43) in paragraph 8.
Previous DNB Working Papers in 2006

No. 81  Arthur van Soest, Arie Kapteyn and Julie Zissimopoulos, Using Stated Preferences Data to Analyze Preferences for Full and Partial Retirement
Financial acceleration of booms and busts