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Dirk Broeders and An Chen *

* Views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.
PENSION BENEFIT SECURITY: A COMPARISON OF SOLVENCY REQUIREMENTS, A PENSION GUARANTEE FUND AND SPONSOR SUPPORT

DIRK BROEDERS‡ AND AN CHEN∗

ABSTRACT. Developed countries apply different security mechanisms in regulation to protect defined pension benefits: solvency requirements, a pension guarantee fund, and sponsor support. We test the performance of these mechanisms in terms of the protection offered to pension benefits in relation to the costs. For this, we calculate the expected log-return for the beneficiaries and the shortfall probability, i.e. the likelihood of the pension payment falling below the promised level. We show that it is possible to compare different pension security mechanisms using appropriate finance tools. Compared to a system based on solvency requirements alone, support by a pension guarantee fund or by the sponsor offers better downside protection for pension funds pursuing an aggressive investment policy. However, this comes at an additional cost. Beneficiaries of pension funds with conservative investment policies are better off under solvency requirements.

Keywords: Pension plans, regulation, barrier options, rainbow barrier options, guarantee systems

JEL: G11, G23

1. Introduction

An occupational pension plan is a financial contract between an employer and its (former) employees aimed at providing a supplementary retirement income. The employer is


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often called the pension plan sponsor and the (former) employees the beneficiaries. The accumulated pension assets and liabilities are in many cases administered by a pension fund. Therefore, pension assets are typically placed bankruptcy remote in a pension fund. Funding a pension plan boils down to accumulating sufficient assets to pay future benefits. A pension plan is fully funded if the value of pension assets is at least equal to that of pension liabilities.

The pension fund’s assets are invested in capital markets and as such exposed to market risks. In addition, the pension fund is exposed to a variety of other risks, including longevity risk, inflation risk, liquidity risk and the sponsor’s default risk. These risks relate to the mismatch of assets and liabilities. If all future cash outflows constituting the pension fund’s liabilities are congruent to the future cash inflows generated by its assets, then mismatch risk is negligible. However, if such a match cannot be realized, then shortfalls or surpluses will occur in the future. The principal duty of pension fund trustees is to keep these risks within acceptable limits so that the beneficiaries’ entitlements are respected. In practice, the mismatch between assets and liabilities is often large. This is typically true for defined benefit pension plans, where beneficiaries receive a minimum guaranteed income after retirement. This amount is usually related to years of service and income. The mismatch risks or funding risks need to be absorbed by one or more of the pension fund’s stakeholders.

This paper, to the best of our knowledge, is the first to rigorously analyze security mechanisms focussing on protecting pension benefits. We consider three security mechanisms to allocate the risks across stakeholders: solvency requirements in which case the pension fund does not share risks with external shareholders but holds additional assets over the liabilities as means of a buffer; a government imposed pension guarantee fund in which the funding risks are born by the pension guarantee fund; and a sponsor guarantee

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1Sometimes, this role may also be performed by an insurance company or directly by the sponsor. If pension assets and liabilities are “on balance sheet” items of the sponsor, the beneficiaries are directly exposed to the sponsor’s default risk.

2The alternative is a defined contribution plan in which the amount of the employer’s annual contribution is specified. In this case the pension depends on the cumulative contributions and investment returns.
in which the funding risks are absorbed by the corporate shareholders.\footnote{Other security mechanisms are contribution adjustments and benefit adjustments. See CEIOPS (2008). In these cases the beneficiaries are the key risk bearers.} This analysis is relevant for the contemporary discussion within the European Union on equivalent regulatory regimes. Historically, the different European countries developed divergent pension systems and consequently diverse regulatory regimes. The European Commission (2010) recently issued a “green paper” on adequate, sustainable and safe pension systems. Part of this common goal is to bridge the gap between different regulatory regimes. The current paper is also relevant for corporations dealing with the funding decisions of their pension plans. Finally, this study may help pension guarantee funds to set a fair and realistic price on their exposures.

This paper begins with describing the solvency margin, the pension guarantee fund and sponsor support in Section 2. Section 3 outlines the pension contract payoff and the economic environment, followed by Section 4 where we analyze the solvency margin. The performance of the pension guarantee fund and the sponsor support are addressed in Section 5. The final section highlights the main conclusions.

2. Different security mechanism

This section focuses on providing some background information of the three security mechanisms: solvency requirements, a pension guarantee fund and sponsor support.

2.1. Solvency requirements. The objective of supervision of pension funds is to offer a high degree of safety to (future) retirees. As a first security mechanism pension funds in some jurisdictions are required to hold a solvency margin of additional assets over the marked-to-market value of pension benefits. This solvency margin is intended to absorb the risks inherent in possible changes in the value of assets and liabilities. The additional assets are also known as “regulatory own funds”.

The additional assets over the liabilities can be used to absorb losses from adverse events on the financial markets or in the development of the liabilities. Typical adverse events include a sharp decline in interest rates, a steep fall in stock prices and an increase in...
longevity estimations. The calculation of the amount of regulatory own funds can, for instance, be based on a Value-at-Risk (VaR) risk measure for a specific time horizon and confidence level.\footnote{Alternatively, Butsic (1994) uses expected policyholder’s deficit as the solvency measure. Olivieri and Pitacco (2003) specifically look at solvency requirements and longevity risk.} This means that theoretically, the required buffer is at least sufficient to prevent a fund’s assets from falling below the level of its liabilities within a certain probability margin. Examples of solvency regimes are found in Denmark (Jørgensen, 2002) and in the Netherlands (Broeders and Pröpper, 2010).

\section*{2.2. Pension guarantee fund.} A pension guarantee fund (PGF) may also operate as the guarantor of defined pension benefits. In this case risks are effectively pooled with other pension funds and corporations. Such a system ensures the promised pension benefits in case the corporation or the pension fund defaults. The PGF charges a premium (or levy) from sponsors or pension funds and pays out pension benefits to beneficiaries whose promises would otherwise be violated. A pension guarantee fund is primarily established to protect a pension fund and its beneficiaries against default risk of its sponsor. On a macro level these idiosyncratic risks can be diversified away by pooling dissimilar firms. This lowers the aggregate costs of protecting against corporate default risks according to the law of large numbers. To control costs, a PGF typically ensures pension liabilities up to a certain maximum.

All the main industrialized countries have a pension guarantee fund. Table 1 shows some key elements of national pension guarantee systems in the U.S., Canada, the U.K., Japan, Germany, Switzerland and Sweden. We distinguish between a PGF’s intervention policy and its contribution policy. Everywhere except Switzerland intervention is triggered by the default of the sponsor. In addition, intervention is escalated if the pension plan is underfunded or has insufficient resources to pay the pension guarantee fund’s contribution. Only in Switzerland is intervention activated if the pension fund itself becomes insolvent. Three possible intervention procedures are distinguished: the assets and liabilities under the pension plan are taken over, annuities are bought or a payment is made to cover the pension fund’s deficit. With respect to contribution policy, it appears that the investment policy does not play a role in any country, while the actual sponsor’s default risk is only considered in the U.K. The degree of underfunding is key in all countries except Germany.
<table>
<thead>
<tr>
<th>Country (Name)</th>
<th>Trigger for intervention</th>
<th>Intervention procedure</th>
<th>Sponsor default risk</th>
<th>Degree of under funding</th>
<th>Pension fund investment</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. (PBGC)</td>
<td>Sponsor default</td>
<td>Assets and liabilities taken over</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Plan underfunded</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Canada (PBGF)</td>
<td>Sponsor default</td>
<td>Payment made to pension fund</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Plan underfunded</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U.K. (PPF)</td>
<td>Sponsor default</td>
<td>Assets and liabilities taken over</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Inability to pay PPF levy</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Japan (PBG)</td>
<td>Sponsor default</td>
<td>Assets and liabilities taken over</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Inability to pay PBG levy</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Germany (PSV)</td>
<td>Sponsor default</td>
<td>Annuities are bought</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Plan underfunded</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sweden (PG)</td>
<td>Sponsor default</td>
<td>Annuities are bought</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Pension fund default</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Switzerland (BVG)</td>
<td>Pension fund default</td>
<td>Annuities are bought</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

TABLE 1. Intervention policy and contribution policy in countries with a pension guarantee fund

Hence the PGF contribution rates are never fully risk-based in any country. This may invoke adverse behavior. A pension fund facing a large deficit may be tempted to take excessive risks and so to gamble for resurrection knowing that the PGF will intervene anyway.

2.3. **Sponsor support.** Apart from through a pension guarantee system, the funding risks may also be shared between a pension fund and the shareholders of the corporation
sponsoring the pension fund. This phenomenon is well known in the literature. Sharpe (1976) was the first to rigorously analyze this relationship in terms of implicit options. If the pension fund’s beneficiaries have a defined benefit and if the corporate shareholders implicitly guarantee this benefit, they have in fact written a put option on the pension fund’s assets. Furthermore, if the shareholders have access to the surplus assets in the pension fund, they have an implicit long position in a call option on the pension fund assets. Other articles, including, Blake (1998) and Broeders (2010), also analyze this kind of implicit options.

Such risk sharing between a pension fund and the sponsor’s shareholders has an economic impact on the former. There is considerable evidence that the funding level of a defined benefit pension plan is reflected in the market value of the sponsor, see, e.g., Feldstein and Seligman (1981), Bulow, Morck and Summers (1987), Carroll and Niehaus (1998) and Coronado and Sharpe (2003). In addition to this value transparency argument, Jin, Merton and Bodie (2006) find that the market risk of the sponsor’s equity reflects the risk level of the pension plan. The economic rationale for this is that the likelihood of a funding shortfall may place a legal – or moral – obligation on the sponsor to increase contributions to the pension fund. On the other hand, a likely surplus in the pension fund tends to be, at least partially, claimed by the sponsor. The values of these contingent claims depend on the volatility of the pension fund’s assets. Furthermore, Rauh (2006) shows that a firm’s capital expenditure declines with mandatory contributions to defined benefit pension plans. This is especially true for underfunded pension funds where the sponsor is legally required to make additional contributions to the fund.

3. Problem Setup

After this general introduction we now turn to an economic analysis of pension benefit security. We consider the pension plan for a single homogeneous group of employees that has to work for another \( T \) years. The homogeneous group can also be considered as a representative beneficiary. Let us assume that at time \( t_0 = 0 \) a conditionally indexed defined benefit pension is issued to a representative beneficiary who provides an upfront contribution \( L \). Such a benefit combines a minimum guaranteed pension income with an
extra return if the pension fund’s assets perform well, and is a hybrid form between defined
benefit and defined contribution, see Broeders (2010) and Broeders and Chen (2010). The
pension fund also receives an initial contribution $S_0$ from the sponsor at time $t_0 = 0$. In
return, the sponsor also has a fair claim on the pension fund’s surplus. This shows that the
sponsor is really sharing risks with the pension fund. Consequently, the initial asset value
$X_0$ of the pension fund is given by the sum of the contributions from both the beneficiary
and the sponsor. From now on, we shall denote $L = \alpha X_0$ with $\alpha \in [0, 1]$. The pension
fund invests the proceeds in a diversified portfolio of risky and non-risky assets.

3.1. **Contract payoff.** We assume that the pension benefits are paid out as a lump sum
at time $T$. The defined benefit can be represented as the initial contributions of the
beneficiaries accumulated with a (nominal) guaranteed rate of return $\delta$: $L_T = L e^{\delta T}$. This
cash flow equals the present value of the annuity payments over the expected remaining
life of the beneficiaries. As such $L_T$ also represents the average of a sequence of cash
flows for different cohorts with an equivalent duration. This assumption is justified by
the observation in practice that pension funds often take the average participant as a
benchmark in decision-making on funding and asset allocation. Usually when the pension
fund’s assets perform well, the beneficiary is entitled to sharing in the surplus. Hence, at
the maturity date $T$, the outstanding liability that the pension fund should redeem to the
beneficiaries is given by

$$
\psi_L(X_T) = L_T + \beta \alpha \left[ X_T - \frac{L_T}{\alpha} \right]^+ - [L_T - X_T]^+
$$

(1)

with $L_T$ being the guaranteed amount at the maturity date $T$. $\beta$ is the participation rate
with which the beneficiaries are allowed to participate in the surpluses of the pension fund.
The payoff given in (1) is the terminal contract payoff if there is no premature liquidation
of the pension fund. It is a combination of a fixed payment $L_T$, a call option on the
pension fund’s assets and a put option on the assets. We allow for the possibility that the

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5The guaranteed rate of return for a pension contract is typically equal to the risk-free rate, so $\delta = r$.
6$L_T$ includes any foreseeable factors over the time to maturity of the contract. E.g., the expected macro
improvement in life expectancy is included in $L_T$. Furthermore, we assume that the pool of beneficiaries is
large enough so that individual mortality risks are fully diversified. There is also no systematic mortality
risk.
pension fund is liquidated before the maturity date, for instance because the pension fund is severely underfunded. In that case a rebate payment is provided to the beneficiaries at the liquidation time $\tau$ immediately. Here we use $\Theta_L(\tau)$ to denote this payoff. To sum up, the beneficiary’s contract payoff consists of two parts: the terminal payment and the rebate payment:

$$\tilde{V}_L := \psi_L(X_T) 1_{\{\tau > T\}} + \Theta_L(\tau)e^{r(T-\tau)} 1_{\{\tau \leq T\}}. \quad (2)$$

For time consistency reasons we assume that the rebate payment accrues at the risk-free rate $r$ over the remaining time to maturity. This is as if the rebate payment is also due at maturity.

3.2. Underlying financial market. Pension funds typically follow a rebalancing strategy in which the actual asset allocation fluctuates closely around a given strategic asset allocation, see, e.g., Bikker, Broeders and de Dreu (2010). To analyze rebalancing, we assume an economy in which there are two traded investment opportunities: a risky and a risk-free asset. The traded risky asset $A$ satisfies

$$dA_t = \mu_A t + \sigma_A W^1_t$$

where $W^1$ is a standard Brownian motion under the market probability measure $P$, i.e. this asset follows Black-Scholes dynamics with an instantaneous rate of return $\mu > 0$ and a constant volatility $\sigma > 0$. Also assume the existence of a riskfree asset $B$ which satisfies

$$dB_t = rB_t dt$$

for a deterministic riskfree rate $r$. The pension fund can only trade in these two assets in a self-financing way starting with initial wealth $x_0$, which is assumed to be larger than the initial contribution level $L$. The wealth process is given by the following stochastic differential equation (SDE)

$$dX_t = X_t(r + \theta(\mu - r))dt + \theta \sigma X_t dW^1_t, \quad X_0 = x_0. \quad (3)$$

Here $\theta$ denotes the fraction of wealth invested in the risky asset $A$ and the remaining invested in the riskless asset $B$. For a given $\theta$, the solution to the SDE is given by

$$X_t(\theta) = X_0 \exp \left\{ \left( r + \theta(\mu - r) - \frac{\theta^2 \sigma^2}{2} \right) t + \sigma \theta W^1_t \right\}. \quad (4)$$
3.3. Problem. In the next section we consider three possible settings, which we briefly introduce here.

- The benchmark case in which the pension fund operates in isolation: it does not share risks with external stakeholders but holds additional assets to absorb funding risks. The benchmark case corresponds with the existence of solvency requirements.
- A guarantee provided by a pension guarantee fund. We assume that the pension fund is required by regulation to make a contract with the pension guarantee fund (PGF). The PGF obtains an upfront premium from the pension fund and will take over the assets and liabilities from the pension fund in case the sponsor goes bankrupt. If the pension fund has a deficit at that point in time, the guarantee fund will become liable for this. By the introduction of a pension guarantee fund, the beneficiaries are apparently ensured with the guaranteed amount, even if the pension fund is underfunded before/at the maturity date. However, the guarantee comes at an upfront premium cost for the pension fund.
- Benefit security provided by the sponsor. The sponsoring corporation covers the pension fund’s deficit fully when its own assets are sufficient to do so. If not, the deficit is covered only partly or not at all.

The key questions of the paper are: what is the expected rate of return the beneficiary achieves on his pension contract under different security mechanisms? What is the probability that the pension beneficiary does not obtain the promised pension in full? Can we draw the conclusion that one security mechanism is superior to another? The answers to these questions are not trivial. With the protections under different securities schemes, the beneficiary may receive a better deal depending on the operative security system. On the other hand, the beneficiary would have to pay additional premiums on top of their initial investment $L$ for the extra protection. Taking the possible additional premium payment into consideration, the expected log-return is defined as

$$R := \frac{1}{T} \log \left( \frac{E[\tilde{V}_L]}{L + \text{additional premium}} \right).$$

Note that payoff $\tilde{V}_L$ takes different values for different security mechanisms. $E$ is the expectation taken under the market probability measure. Furthermore, we assume there is no additional premium payment in the case of regulatory own funds. The shortfall
probability can be defined by

\[ SP := E[1_{\{X_T \leq L_T\}}1_{\{\tau > T\}}] + E[1_{\{X_\tau \leq L_\tau\}}1_{\{\tau \leq T\}}]. \]  

(6)

This shortfall probability consists of two parts: if the pension fund is still operating at time \( T \), the shortfall occurs when the assets \( X_T \) of the pension fund are insufficient to pay the promised pension benefits \( L_T = L e^{\delta T} \). If the pension fund stops operating prematurely at \( \tau \leq T \), the shortfall occurs when \( X_\tau < L_\tau = L e^{\delta \tau} \). The default time \( \tau \) is specified in each security mechanism. Throughout the paper we calculate the shortfall probability under the market probability measure \( P \). However, the insurance premiums are determined as the expected discounted payoff under the unique risk-neutral probability \( P^* \).

4. Benchmark model: solvency requirements

In the case of solvency requirements we neglect all other external security mechanisms at a premature default. We model the premature default in a standard barrier option framework as in Grosen and Jørgensen (2002). The default-triggering event is the underfunding of the pension fund. The intervention time, where the regulator steps in and liquidates the pension fund, is the first time the pension fund’s assets breach the regulatory threshold:

\[ \tau_p := \inf\{t \mid X_t \leq \eta L e^{\delta t}\} \]  

(7)

Hereby we assume \( \eta \leq 1 \).\(^7\) If \( \tau_p \leq T \), the pension fund is liquidated immediately at \( \tau_p \).\(^8\) At the default time, the pension fund provides what remains in the pension fund to the beneficiary, i.e. \( \eta \) fraction of the guaranteed amount. Hence, the rebate payment to the beneficiary is

\[ \Theta_L(\tau_p) = X_{\tau_p} = \eta L e^{\delta \tau_p} \]  

(8)

Furthermore, it is assumed that initially \( X_0 > \eta L \), otherwise the pension fund is already defaulted at inception.

\(^7\)In practice the regulatory boundary is often at or close to 1. However, if the funding ratio is below this level this will not automatically entail liquidation of the pension fund, as a grace period is often allowed for recovery. Therefore, the effective barrier level is lower. See Broeders and Chen (2010) for a formal analysis of the impact of recovery periods.

\(^8\)We hereby implicitly assume continuous monitoring and prompt corrective action by the regulator.
We are interested in computing the expected rate of return $R$ on the beneficiaries’ initial investment. The mean terminal payoff for the beneficiary is given by the sum of the expected payment at maturity and the expected payment beforehand

$$E[\hat{V}_L] = E[\psi_L(X_T) 1_{\{\tau_p>T\}} + \Theta_L(\tau_p) e^{r(\tau_p-T)} 1_{\{\tau_p\leq T\}}]$$

$$= E[(L_T + \beta \alpha [X_T - L_T/\alpha]^+ - [L_T - X_T]^+) 1_{\{\tau_p>T\}}] + E[\eta Le^{\delta \tau_p} e^{r(\tau_p-T)} 1_{\{\tau_p\leq T\}}].$$

(9)

All the four components of $E[\hat{V}_L]$ are given explicitly in Propositions 4.1 and 4.2.

**Proposition 4.1.** In the benchmark case, the expected fixed payment is:

$$E[L_T 1_{\{\tau_p>T\}}] = L_T N \left( d^- (X_0, B_0 e^{-(\delta-\tilde{\mu})T}) \right) - L_T \left( \frac{X_0}{B_0} \right) \frac{-2(\check{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N \left( d^+ (B_0^2, X_0 e^{-\tilde{\mu} T}) \right)$$

(10)

with $d^\pm (S, K) = \frac{\ln(S/K) + \frac{1}{2} \theta^2 \sigma^2 T}{\theta \sigma \sqrt{T}}$, $B_0 = \eta L$ and $\tilde{\mu} = r + \theta (\mu - r)$. The expected bonus call option can be calculated as follows:

$$E \left[ X_T - \frac{L_T}{\alpha} \right]^+ 1_{\{\tau_p>T\}} = X_0 e^{\tilde{\mu} T} N \left( d^+ (X_0, Y e^{-\tilde{\mu} T}) \right) - L_T \left( \frac{X_0}{B_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N \left( d^+ ((B_0)^2, X_0 Y e^{-\tilde{\mu} T}) \right)$$

$$- X_0 e^{\tilde{\mu} T} \left( \frac{B_0}{X_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N \left( d^- ((B_0)^2, X_0 Y e^{-\tilde{\mu} T}) \right)$$

$$+ \frac{L_T}{\alpha} \left( \frac{B_0}{X_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N \left( d^- ((B_0)^2, X_0 Y e^{-\tilde{\mu} T}) \right)$$

(11)

with $Y := \max\{B_T, \frac{L_T}{\alpha}\}$. The down–and–out put option is then determined by

$$E[[L_T - X_T]^+ 1_{\{\tau_p>T\}}]$$

$$= L_T \left[ N(-d^- (X_0, L_T e^{-\tilde{\mu} T})) - N(d^+ (B_0, X_0 e^{(\tilde{\mu} - \delta)T})) \right]$$

$$- \left( \frac{B_0}{X_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} \cdot N(d^- (B_0, X_0 e^{(\tilde{\mu} - \delta)T})) + \left( \frac{B_0}{X_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N(d^- (B_0^2, X_0 L_T e^{-\tilde{\mu} T}))$$

$$- X_0 e^{\tilde{\mu} T} \left[ N(-d^+ (X_0, L_T e^{-\tilde{\mu} T})) - N(-d^+ (X_0 e^{(\tilde{\mu} - \delta)T}, B_0)) \right]$$

$$- \left( \frac{B_0}{X_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N(d^+ (B_0, X_0 e^{(\tilde{\mu} - \delta)T})) + \left( \frac{B_0}{X_0} \right) \frac{2(\tilde{\delta} - \frac{1}{2} \theta^2 \sigma^2)}{\theta^2 \sigma^2} N(d^+ (B_0^2, X_0 L_T e^{-\tilde{\mu} T}))$$

(12)
Proof: Similar calculations have been carried out in Broeders, Chen and Koos (2009). In comparison with the calculations in that paper, we need to make several replacements. We replace $A_0$ by $X_0$, $\mu$ by $\tilde{\mu}$, $\tau$ by $\tau_p$, $r$ by $\delta$ and finally $\sigma$ by $\theta \sigma$.

Proposition 4.2. The expected accumulated rebate payment to the beneficiary can be represented by

$$E[\eta L e^{r(T-\tau_p)} 1_{\{\tau_p \leq T\}}] = \eta L \int_0^T e^{\delta s} e^{r(T-s)} \tilde{f}(s) ds$$

(13)

where $\tilde{f}(s)$ is the density of the first hitting time $\tau_p$ under $P$:

$$\tilde{f}(s) = - \frac{\ln \frac{\eta L}{X_0}}{\sigma \theta s^{3/2}} n \left( \frac{\ln \frac{\eta L}{X_0} - \tilde{\mu} s}{\sigma \theta \sqrt{s}} \right).$$

(14)

with $\tilde{\mu} = \tilde{\mu} - \delta - \frac{\sigma^2 \theta^2}{2}$ and $n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, the density of a standard normally distributed random variable.

Proof: For any finite $s$, we have under $P$

$$P(\tau_p \leq s) = N \left( \frac{\ln \left( \frac{\eta L}{X_0} \right) - \tilde{\mu} s}{\delta \sigma \sqrt{s}} \right) + \left( \frac{X_0}{\eta L} \right) \frac{2 \tilde{\mu}}{\sigma^2 \sigma^2} \sigma \theta \sqrt{\frac{n \left( \ln \left( \frac{\eta L}{X_0} \right) \right) \frac{\eta L}{X_0} - \tilde{\mu} s}{\sigma \theta \sqrt{s}}}.$$

The density is then given by taking the first derivative with respect to $s$. Using the equality:

$$n \left( \ln \left( \frac{\eta L}{X_0} \right) \right) = \left( \frac{X_0}{\eta L} \right) \frac{2 \tilde{\mu}}{\sigma^2 \sigma^2} \sigma \theta \sqrt{\frac{n \left( \ln \left( \frac{\eta L}{X_0} \right) \right) \frac{\eta L}{X_0} - \tilde{\mu} s}{\sigma \theta \sqrt{s}}}.$$

yields the density given in (14).

The integral in proposition 4.2 can be calculated numerically. In Table 2, the expected log-return is calculated for different participation rates $\beta$ and different allocations to the risky asset $\theta$. We observe the following. First, it is evident that the beneficiary benefits from a higher participation in possible surpluses. Hence, a higher $\beta$ leads to a higher expected rate of return. Second, the effect of $\theta$ is manifold and ambiguous. An increase in $\theta$ leads to a higher default probability for the pension fund and, by consequence, an increased expected rebate payment. On the other hand, a higher $\theta$ might make the expected terminal payment either rise or drop. Hence, there is a non-monotonic effect of $\theta$ on the expected log-return. If the exposure to the risky asset is very high, we might obtain an expected log return smaller than the risk-free rate. The reason for this is that the beneficiary is fully exposed to downside risk but has only limited upside potential as the pension is capped at full indexation. Finally, it is remarkable to note that in some
\[ \beta = 0.6 \quad \beta = 0.7 \quad \beta = 0.8 \quad \beta = 0.9 \]

<table>
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<th>( \beta = 0.8 )</th>
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</tr>
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<td>( \theta = 0.9 )</td>
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<td>0.0433</td>
<td>0.0463</td>
<td>0.0492</td>
</tr>
</tbody>
</table>

**Table 2.** The expected log-return under solvency requirements as a function of the participation rate \( \beta \) and the allocation to risky assets \( \theta \) with parameters: \( \mu = 0.08, \sigma = 0.2, T = 15, X_0 = 100, L = 90, \delta = r = 0.05, \eta = 0.9 \).

In cases, the beneficiary achieves a rate of return below the rate from investing in the risk-free asset only. This is plausible in our setting. For \( \eta < 1 \) the rebate payment is smaller than the promised pension payment and when there is no premature termination, the payoff of the beneficiary is not necessarily larger than the guaranteed amount. Next we turn to the computation of the shortfall probability.\(^9\)

**Proposition 4.3.** In the benchmark model, the shortfall probability defined in (6) is given by:

\[
SP = 1 - N \left( d^- \left( X_0, Y e^{-\tilde{\mu} T} \right) + \left( \frac{B_0}{X_0} \right) \left( \mu - \frac{1}{2} \delta^2 \sigma^2 \right) \right) N \left( d^- \left( \left( B_0 \right)^2, X_0 Y e^{-\tilde{\mu} T} \right) \right). \tag{15}
\]

**PROOF:** The shortfall probability is given by:

\[
E[1_{\{X_T \leq L_T \}}1_{\{\tau_p > T\}}] + E[1_{\{X_{\tau p} \leq L_{\tau p} \}}1_{\{\tau_p \leq T\}}]
\]

\[
= P(\tau_p > T) - P(X_T > L_T, \tau_p > T) + P(\tau_p \leq T)
\]

\[
= 1 - P(X_T > L_T, \tau_p > T)
\]

\[
= 1 - N \left( d^- \left( X_0, Y e^{-\tilde{\mu} T} \right) + \left( \frac{B_0}{X_0} \right) \left( \mu - \frac{1}{2} \delta^2 \sigma^2 \right) \right) N \left( d^- \left( \left( B_0 \right)^2, X_0 Y e^{-\tilde{\mu} T} \right) \right)
\]

\(^9\)Loss given default is another measure of risk. We leave this for future research.
with $Y = \max\{B_T, L_T\}$. From step 1 to 2 we have used the fact that $X_{\tau_p} = \eta L_{\tau_p} \leq L_{\tau_p}$. The last step can be derived by using the reflection principle of the Brownian motion and the result can be read from Proposition 4.1.

Note that the shortfall probability depends closely on the barrier level $\eta$ and the allocation to the risky asset as shown in Table 3. The higher the barrier, the more likely the pension fund is to default and thereby suffer a shortfall. Upon default, the pension assets are insufficient to cover the promised pension payment $L_{\tau_p}$ because in practice the barrier level will almost always be smaller than $L_{\tau_p}$. The effect of the barrier level $\eta$ is, in fact, even magnified when it is combined with the equity exposure. The higher the equity allocation $\theta$, the more likely it is that a shortfall will occur. For extremely high equity holdings, e.g. $\theta = 0.9$, the shortfall occurs with a probability of 84.7% for $\eta = 1$. However, as the probabilities appear relatively high, this may raise doubts about their viability. Here, note that our analysis is conducted for a 15-year time horizon. A shortfall probability of 84.7% for a 15-year time span is roughly equivalent to a shortfall probability of $12.0\% \approx 1 - (1 - 84.7\%)^{1/15}$ over a one-year horizon.

\[\begin{array}{ccccccc}
\hline
& \eta = 0.6 & \eta = 0.7 & \eta = 0.8 & \eta = 0.9 & \eta = 1 \\
\hline
\theta = 0.1 & 0.0286 & 0.0286 & 0.0286 & 0.0286 & 0.0758 \\
\theta = 0.3 & 0.1793 & 0.1794 & 0.1868 & 0.2590 & 0.5103 \\
\theta = 0.5 & 0.2565 & 0.2716 & 0.3325 & 0.4740 & 0.6970 \\
\theta = 0.7 & 0.3261 & 0.3749 & 0.4717 & 0.6155 & 0.7911 \\
\theta = 0.9 & 0.4058 & 0.4779 & 0.5821 & 0.7091 & 0.8469 \\
\hline
\end{array}\]

Table 3. Shortfall probability under the solvency regime as a function of barrier level $\eta$ and allocation to risky assets $\theta$ with parameters: $\mu = 0.08$, $\sigma = 0.2$, $T = 15$, $X_0 = 100$, $L = 90$, $\delta = r = 0.05$.

Note that if the pension fund were to invest a hundred percent in risk-free assets, the probability of a default would be nil. However, in that case the pension fund would also not be able to offer any indexation.
5. Analyzing other security mechanisms

Under solvency requirements only, the beneficiary is not ensured the promised pension benefits. Conceptually, one cannot write a guarantee to oneself. The shortfall, i.e. the case where beneficiaries receive an amount below the promised pension payment, occurs with a certain likelihood. For a true guarantee the pension fund needs an external guarantor. In order to protect beneficiaries by ensuring them the promised pension payments (both upon the sponsor’s or the pension fund’s premature default or upon their survival of the retirement date), jurisdictions may impose such external security mechanisms on the pension funds. As discussed before, we consider two mechanisms: a guarantee provided by a pension guarantee fund and a sponsor guarantee. The former provides a full guarantee to the beneficiary while the latter may entail a full/partial/no guarantee to the beneficiary, depending on the financial situation of the sponsor.

5.1. Pension guarantee fund. We assume that the pension fund closes a contract with the PGF. The PGF receives an upfront premium from the pension fund and will take over its assets and liabilities should the corporation default.\footnote{In reality, periodic premiums are charged by the pension guarantee fund. Here a single premium is assumed for simplicity and for consistency with single liability for the representative beneficiary. Implicitly this premium payment is paid by the beneficiaries as part of their total labor compensation.} If the pension fund has a deficit at that point in time, the guarantee fund will be liable for this. In line with most countries, we assume the intervention procedure is based on the sponsor’s default. Therefore we need to introduce a stochastic price process for the sponsor’s assets. Assume that these evolve over time according to

\[
dC_t = \mu_c C_t dt + \sigma_c C_t (\rho dW^1_t + \sqrt{1-\rho} dW^2_t)
\]

with instantaneous rate of return \(\mu_c > 0\) and volatility \(\sigma_c > 0\). Here \(W^1_t\) and \(W^2_t\) are two independent Brownian motions under the probability measure \(P\). The corporate’s asset correlates with the risky asset \(A_t\) with a correlation coefficient \(\rho\). Define the sponsors’ default time \(\tau_c\) as

\[
\tau_c = \inf \{t | C_t \leq \phi C_0 e^{\gamma t} \}.
\]
The threshold level for the corporation to default is its debt level $\phi C_0$, $\phi \in (0, 1)$, where $\phi$ reflects the leverage level of the corporation. Note that $\phi$ needs to be smaller than 1, otherwise the corporation already defaults at inception. $\phi C_0$ can be considered as the initial debt level and for simplicity reasons, we assume $\phi$ is a constant. However, we also allow for the possibility that the debt level increases over time with a constant growth rate $g$. If the corporate’s assets hit the threshold before the retirement date $T$, the corporation defaults instantaneously. At this time, the PGF intervenes and makes any necessary payments to the pension fund.

Note that whether or not the guarantee fund takes over the pension fund depends on the corporate’s default. We model the insurance provided by the pension guarantee fund by distinguishing between two cases: no premature default of the sponsoring company, i.e. $\tau_c > T$, and premature default of the sponsoring company, i.e. $\tau_c \leq T$. In the former case, the PGF needs to cover the deficit of the pension fund, if any. Hence, the payoff of the pension insurance at maturity is

$$G_T = \max(Le^{\delta T} - X_T, 0).$$

In the latter case, the PGF intervenes immediately at $\tau_c$ that is, it takes over the pension fund and the operation of the pension fund is terminated. At any $t < \tau_c$, the pension fund trades as usual in the risky asset $A$ and the risk-free asset $B$ in a self-financing way starting with initial wealth $x_0$ as in (3). Recall that the PGF’s responsibility is to cover the deficit of the pension fund, if any. Assume that the PGF needs to cover the deficit immediately at $\tau_c$. Therefore, the premature payoff of the pension insurance is

$$G_{\tau_c} = \max(Le^{\delta \tau_c} - X_{\tau_c}, 0).$$

Upon premature termination of the pension plan, the pension guarantee fund covers the deficit if the pension fund’s assets at that time are insufficient to pay the target guarantee $Le^{\delta \tau_c}$. More compactly, the cost $G$ of the pension insurance provided by the PGF is given

12In reality, the PGF’s insurance payment is capped. The PBGC insurance program pays pension benefits up to the maximum guaranteed benefit set by law to participants who retire at age 65.
The cost of insurance can be decomposed into two parts: a rainbow down-and-out put option and a rainbow down-and-in put option. Rainbow barrier options are a well-known form of barrier options where the option is written on one underlying asset while the knock-in or knock-out condition is triggered by a second asset.\textsuperscript{13} In the case of pension insurance, the underlying asset is formed by the pension fund’s assets and the knock-out is triggered by the plan sponsor’s assets. Upon default of the sponsor, the pension fund is taken over by the PGF. We have a rainbow down-and-out put option if there is no premature termination of the pension fund, and a rainbow down-and-in put option if there is premature termination at $\tau_c$.

5.1.1. \textit{Premium to the pension guarantee fund, expected log-return}. According to the definition in (5), we first need to know the additional premium the pension fund needs to pay to the PGF in order to receive the guarantee.\textsuperscript{14} Next, we may determine the expected payoff of the contract and the expected rate of return. We start by calculating the premium. Unlike in the analysis above – which is conducted under the market measure $P$ – the premium calculation is carried out under the risk-neutral measure $P^*$ which is defined by the following Radon-Nikodym density:

$$ \frac{dP^*}{dP} |_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \lambda_1 dW_t^1 - \int_0^T \lambda_2 dW_t^2 - \frac{1}{2} \left( \int_0^T \lambda_1^2 dt - \int_0^T \lambda_2^2 dt \right) \right\} $$

\[\begin{align*}
\lambda_1 &= \frac{\mu - r}{\sigma} \\
\lambda_2 &= \frac{\mu_c - r}{\sigma_c \sqrt{1 - \rho^2}} - \frac{\rho}{\sqrt{1 - \rho^2}} \frac{\mu - r}{\sigma},
\end{align*}\]

\textsuperscript{13}These exotic options were firstly analyzed in Heynen and Kat (1994), Zhang (1995) and Carr (1996).

\textsuperscript{14}The determination of the appropriate premium has been discussed in other papers, e.g., Marcus (1987), Chen, Ferris and Hsieh (1994), Lewis and Pennacchi (1994) and Boyce and Ippolito (2002). We shall point out that our insurance payoff of the PGF differs from the others because of the incorporated default mechanism.
i.e. $dW^{*1}_t = dW^1_t + \lambda_1 dt$ and $dW^{*2}_t = dW^2_t + \lambda_2 dt$. Under the risk-neutral measure, the corporate assets and the pension fund’s portfolio evolve as follows

$$dC_t = rC_t dt + \sigma C_t (\rho dW^{*1}_t + \sqrt{1-\rho^2} dW^{*2}_t)$$

$$dX_t = rX_t dt + \theta \sigma X_t dW^{*1}_t.$$ 

The solution to these SDEs are

$$C_t = C_0 \exp \left\{ \left( r - \frac{1}{2} \sigma_c^2 \right) t + \sigma_c (\rho W^{*1}_t + \sqrt{1-\rho^2} W^{*2}_t) \right\}$$

$$X_t = X_0 \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma \theta W^{*1}_t \right\}.$$ 

According to the fair contract principle, the premium paid by the pension fund to the PGF should equal the expected discounted payment (under the risk neutral measure) the PGF provides. Therefore, the premium is determined by

$$G_0 = E^* [e^{-rT} G_T 1_{\{\tau_c > T\}}] + E^* [e^{-r\tau_c} G_{\tau_c} 1_{\{\tau_c \leq T\}}]$$

$$= E^* [e^{-rT} [L e^{\delta T} - X_T] 1_{\{\tau_c > T\}}] + E^* [e^{-r\tau_c} [L e^{\delta \tau_c} - X_{\tau_c}] 1_{\{\tau_c \leq T\}}],$$

where $E^*$ denotes the expected value under the risk-neutral measure $P^*$.

**Proposition 5.1.** The fair premium paid to the pension guarantee fund consists of two parts: the market value of the rainbow down-and-out put option and the market value of the rainbow down-and-in put option. The price of the rainbow down-and-out put option maturing at the retirement date $T$:

$$E^* [e^{-rT} \max(L e^{\delta T} - X_T, 0) 1_{\{\tau_c > T\}}]$$

$$= -X_0 \left[ M (-d_1, -e_1; -\rho) - e^{2 \left( r-g - \frac{1}{2} \sigma^2 + \rho \theta \sigma_c \right) \ln(\phi C_0/C_0)} M (-d_3, -e_3; -\rho) \right]$$

$$+ L e^{\delta T} e^{-rT} \left[ M (-d_2, -e_2; -\rho) - e^{2 \left( r-g - \frac{1}{2} \sigma^2 + \rho \theta \sigma_c \right) \ln(\phi C_0/C_0)} M (-d_4, -e_4; -\rho) \right]$$
where
\[
\begin{align*}
d_1 &= \ln(X_0/(Le^{\delta T})) + (r + \frac{1}{2}\theta^2\sigma^2) T \\
d_2 &= d_1 - \sigma\theta\sqrt{T} \\
d_3 &= d_1 + \frac{2\rho\ln(\phi C_0/C_0)}{\sigma_c\sqrt{T}} \\
d_4 &= d_2 + \frac{2\rho\ln(\phi C_0/C_0)}{\sigma_c\sqrt{T}} \\
e_1 &= \ln(\phi C_0/C_0) - (r - g - \frac{1}{2}\sigma^2_c + \rho\sigma\theta\sigma_c) T \\
e_2 &= e_1 + \rho\sigma\theta\sqrt{T} \\
e_3 &= e_1 - \frac{2\ln(\phi C_0/C_0)}{\sigma_c\sqrt{T}} \\
e_4 &= e_2 - \frac{2\ln(\phi C_0/C_0)}{\sigma_c\sqrt{T}}
\end{align*}
\]

with \(M(a, b; \rho)\) denoting the cumulative bivariate normal distribution function defined as
\[
M(a, b; \rho) = \frac{1}{2\pi} \int_a^{-\infty} \int_b^{-\infty} \exp\left\{\frac{-x^2 - 2\rho xy + y^2}{2(1 - \rho^2)}\right\} dx dy.
\]

The price of the rainbow dow-and-in put option can be further expressed as:
\[
E^*\left[e^{-r\tau_c} \max(Le^{\delta \tau_c} - X_{\tau_c}, 0) 1_{\{\tau_c \leq T\}}\right] = \int_0^T e^{-rs} \int_{-\infty}^{d(s)/\sqrt{s}} \left(Le^{\delta s} - X_0 \exp\left\{\left(r - \frac{1}{2}\theta^2\sigma^2\right) s + \theta\sigma\left(\frac{\rho}{\sigma_c} (\ln \phi - \hat{r}_c s) + \sqrt{1 - \rho^2} \sqrt{s x}\right)\right\}\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx f(s) ds
\]

\[
d(s) = \frac{\ln \left(\frac{Le^{\delta s}}{X_0}\right) - \frac{\theta\sigma}{\sigma_c} \ln \phi - \left(r - \frac{1}{2}\theta^2\sigma^2 - \frac{\theta\sigma}{\sigma_c} (r - g - \frac{1}{2}\sigma^2_c)\right) s}{\theta\sigma\sqrt{1 - \rho^2}}
\]

where \(\hat{r}_c = r - g - \frac{1}{2}\sigma^2_c\) and \(f(s)\) is density of the first hitting time
\[
f(s) = \frac{-\ln \phi}{\sigma_c s^{3/2}} h\left(\frac{\ln \phi - \hat{r}_c s}{\sigma_c \sqrt{s}}\right).
\]

**Proof:** The proof can be found in Appendix 6.2. ■

The integral for the premium calculation can be approximated numerically. Table 4 provides the outcomes for the premiums. There are several observations. First, the premium falls when the corporate leverage ratio \(\phi\) is higher or when \(g\) is higher. This might seem counterintuitive but is the result of two opposite effects. On the one hand, it is due to the fact that a higher \(\phi\) or \(g\) each lead to a higher default barrier, which indicates that the corporate is more likely to default triggering the PGF’s intervention to cover a potential deficit. On the other hand, early default also implies that the conditional expected shortfall at the pension fund level is low. In other words, the rainbow-down-and-in put dominates in the premium calculation. Since here the rainbow-down-and-in put has
Table 4. Fair levies charged by the pension guarantee fund as a function of the sponsor’s leverage ratio $\phi$, debt growth rate $g$ and the correlation coefficient $\rho$ between the sponsor’s and the pension fund’s assets, with parameters: $T = 15$, $r = 0.05$, $\sigma_c = 0.33$, $\sigma = 0.2$, $\theta = 0.5$, $X_0 = 100$, $L = 90$ and $\delta = 0.05$.

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<th>$g$</th>
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</table>

a lower value than the rainbow-down-and-out put, we observe a negative relation between the premium and $\phi$ or $g$. A second observation is that the premium increases with the correlation coefficient $\rho$. A negative correlation between the sponsor’s and the pension fund’s assets results in a lower premium. If there is a negative correlation, the pension fund’s may be doing very well at the time the corporate defaults. By consequence, the chance that guarantee fund will have to cover the deficit is smaller, necessitating a lower premium.
Next we turn to the calculation of the expected log-return in the pension guarantee fund case. Under the PGF security scheme, the beneficiaries’ expected payment is given by

\[
E[(\psi_L(X_T) + G_T)1_{\{\tau_c \geq T\}}] + E[(\Theta_L(\tau_c) + G(\tau_c))e^{r(T-\tau_c)}1_{\{\tau_c \leq T\}}]
\]

\[
= E\left( L_T + \beta \alpha \left[ X_T - \frac{L_T}{\alpha} \right]^+ 1_{\{\tau_c > T\}} \right) + E[Le^{\delta\tau_c}e^{r(T-\tau_c)}1_{\{\tau_c \leq T\}}]
\]  \hspace{1cm} (19)

Under this security mechanism, the beneficiary’s payoff is raised by the insurance guarantee provided by the PGF. For \( \tau_c > T \), the guarantee increases the payoff of the beneficiary by \( [Le^{\delta T} - X_T]^+ \) (compared to the payoff under a solvency regime). That means that the beneficiary is ensured with at least \( L_T \), and possibly a participation in the bonus. For \( \tau_c \leq T \), through the guarantee of the PGF, the beneficiary is ensured with the payment \( Le^{\delta\tau_c} \) at \( \tau_c \), which we have accumulated with the risk-free rate until the maturity date \( T \). When calculating the expected rate of return \( R \), the denominator needs to be adjusted to \( L + G_0 \). The expected payoff can be decomposed into three terms: the expected terminal guaranteed amount \( E[L_T1_{\{\tau_c > T\}}] \); a rainbow down-and-out call option \( \beta \alpha E \left[ \left[ X_T - \frac{L_T}{\alpha} \right]^+ 1_{\{\tau_c > T\}} \right] \); and the expected rebate payment \( E[Le^{\delta\tau_c}e^{r(T-\tau_c)}1_{\{\tau_c \leq T\}}] \).

**Proposition 5.2.** The expected terminal guaranteed amount can be easily determined, because the survival probability is equal to 1 minus the default probability \( P(\tau_c \leq T) \):

\[
P(\tau_c \leq T) = P\left( \inf_{t \in [0,T]} C_t \leq \phi C_0 e^{gt} \right)
= N\left( \ln(\phi) - \mu_c T \sigma_c \right) + \left( \frac{1}{\phi} \right) N\left( \frac{\ln(\phi) + \mu_c T}{\sigma_c \sqrt{T}} \right)^{-\frac{2\mu_c}{\sigma^2}}
\]

\hspace{1cm} (20)

with \( \mu_c = \mu_c - g - \frac{\sigma^2}{2} \).

**Proof:** Again the probability can be found calculated explicitly in the literature and the proof can be found in, e.g., Haug (2007).

**Proposition 5.3.** The expected rainbow down-and-out call option in (19)
\( \left( \beta \alpha E \left[ \left[ X_T - \frac{L_T}{\alpha} \right]^+ 1_{\{\tau_c > T\}} \right] \right) \) can be expressed in terms of the cumulative bivariate normal distribution function with correlation coefficient \( \rho \) as follows (see Haug (2007)):
\[ RBO = \beta \alpha X_0 e^{\bar{\mu} T} \left[ M(d_1, -e_1; \rho) - e^{\frac{2(\mu_c - g - \frac{1}{2} \sigma^2_c + \rho \sigma_c \sigma_e)}{\sigma^2_e} \ln(C_0/C_0)} M(d_3, -e_3; \rho) \right] \]

\[ - \beta L_T \left[ M(d_2, -e_2; \rho) - e^{\frac{2(\mu_c - g - \frac{1}{2} \sigma^2_c)}{\sigma^2_e} \ln(C_0/C_0)} M(d_4, -e_4; \rho) \right] \]

where \( \bar{\mu} = r + \theta (\mu - r) \) and 

\[
\begin{align*}
    d_1 &= \frac{\ln(X_0/(L_T/\alpha)) + (\bar{\mu} + \frac{1}{2} \theta^2 \sigma^2)}{\sigma \theta \sqrt{T}} \\
    d_2 &= d_1 - \sigma \theta \sqrt{T} \\
    d_3 &= d_1 + \frac{2 \rho \ln(C_0/C_0)}{\sigma \sqrt{T}} \\
    d_4 &= d_2 + \frac{2 \rho \ln(C_0/C_0)}{\sigma \sqrt{T}} \\
    e_1 &= \frac{\ln(C_0/C_0) - (\mu_c - g - \frac{1}{2} \sigma^2_c + \rho \sigma_c \sigma_e) T}{\sigma_c \sqrt{T}} \\
    e_2 &= e_1 + \rho \sigma \theta \sqrt{T} \\
    e_3 &= e_1 - \frac{2 \ln(C_0/C_0) \sigma_c \sqrt{T}}{\sigma_c \sqrt{T}} \\
    e_4 &= e_2 - \frac{2 \ln(C_0/C_0)}{\sigma_c \sqrt{T}}.
\end{align*}
\]

**Proof:** The proof of this proposition is very similar to the proof of Proposition 5.1 given in Appendix 6.2. Therefore it is not given here in the interest of brevity.

**Proposition 5.4.** The expected accumulated rebate payment of the beneficiaries can be determined by 

\[ E[Le^{\delta \tau_c}e^{r(T-\tau_c)}1_{\{\tau_c<T\}}] = L \int_0^T e^{\delta s} e^{r(T-s)} \hat{f}(s) ds \]  

(21)

where \( \hat{f}(s) \) is the density of the first hitting time \( \tau_c \) under \( P \):

\[ \hat{f}(s) = -\frac{\ln \frac{C_0}{C_0}}{\sigma_c s^{3/2}} n \left( \frac{\ln \frac{C_0}{C_0} - \mu_c s}{\sigma_c \sqrt{s}} \right), \]  

(22)

\[ \mu_c = \mu_c - g - \frac{\sigma^2}{2}. \]

**Proof:** The proof of this proposition is very similar to the proof of Proposition 4.2. Therefore, in the interest of brevity it is omitted here.

In Table 5, the expected log-return for the case of a pension guarantee fund is computed for different combinations of \( \theta \) and \( \beta \). We observe the following. First, similar to the benchmark case with regulatory own funds, a higher participation rate \( \beta \) indicates a higher expected rate of return for the beneficiary, whereas the effect of \( \theta \) appears different. The more the pension fund invests in the risky asset, the higher the expected rate of return.
Table 5. The expected rate of return under a regime with a pension guarantee fund as a function of the participation rate $\beta$ and allocation to risky assets $\theta$ with parameters: $T = 15$, $r = 0.05$, $\mu_c = 0.1$, $\sigma_c = 0.33$, $\mu = 0.08$, $\sigma = 0.2$, $\theta = 0.5$, $X_0 = 100$, $L = 90$, $\phi = 0.8$, $\eta = 1$, $g = 0.01$, $\rho = 0.5$ and $\delta = 0.05$.

Intuitively speaking, since the beneficiary has now the absolute guarantee provided by the PGF, the beneficiaries can benefit more from risky investments because they do not really suffer from the downside risk. However, they pay additional contributions for the downside protection. Second, for those pension funds which invest very conservatively (the holding of the risky asset is low), the resulting expected log-return is lower than under regulatory own funds, whereas for those pension funds which invest very aggressively (the holding of the risky asset is high), the expected log-return is larger than under regulatory own funds. In other words, the insurance of the PGF actually makes the beneficiary worse off for pension funds with conservative investment policies. A conservative investment policy makes the insurance quite unattractive because of the relatively high premium payments.

As observed in Figure 1, the expected log-return also strongly depends on the correlation between the pension fund’s and the sponsor’s assets. If the two assets are weakly correlated, the log-return is relatively small. In addition, it is interesting to note that there is a negative relation between the sponsor’s leverage ratio $\phi$ and the log-return. When $\phi$ is set lower, a lower corporate default barrier level results. In other words, the pension guarantee fund is less likely to intervene, lower premiums need to be paid and consequently the beneficiary achieves a higher expected log-return.
Figure 1. The expected log-return as a function of the correlation coefficient $\rho$ between the sponsor’s and the pension fund’s assets for different $\phi$’s with parameters: $T = 15$, $r = 0.05$, $\mu_c = 0.1$, $\sigma_c = 0.33$, $\mu = 0.08$, $\sigma = 0.2$, $\theta = 0.5$, $X_0 = 100$, $L = 90$, $\beta = 0.8$ and $\delta = 0.05$.

Since the PGF always ensures the beneficiary the payment of $L_{\tau_c}$ at $\tau_c$ or of $L_T$ at $T$ (whichever is lower), there is no shortfall risk for the beneficiary.

5.2. Sponsor guarantee. This subsection deals with the case where the sponsor offers protection against the premature default of the pension fund. As in the benchmark case, the triggering event is underfunding of the pension fund and the intervention time is the first time the pension fund’s assets breach the regulatory threshold:

$$\tau_p = \inf\{t | X_t \leq \eta L e^{\delta t}\}.$$

If the pension fund defaults at $\tau_p$, it holds assets worth $X_{\tau_p} = \eta L_{\tau} \leq L_{\tau_p}$. For $\eta = 1$, the firm does not need to provide any guarantee, because the asset value of the pension fund at $\tau_p$ is exactly the promised pension payment. Hence, we assume here $\eta < 1$ for this part of analysis. For $\eta < 1$, if there is premature default of the pension fund, the firm has to cover the deficit given that

a) covering the pension fund’s deficit does not lead to the default of the sponsor;
b) if the sponsor has not defaulted but is unable to cover the entire deficit \( L e^{\delta \tau_p} - X_{\tau_p} \),

it pays what it still has after paying back the corporate debt. In other words, the
sponsor aims to pay back its corporate debt first.

According to these descriptions, the financial support that the sponsoring corporate
needs to provide at time \( \tau_p \leq T \) is described by

\[
\Phi_c(\tau_p) = (L e^{\delta \tau_p} - X_{\tau_p}) 1_{\{C_{\tau_p} > \phi C_0 e^{\eta \tau_p} + (L e^{\delta \tau_p} - X_{\tau_p})\}}
+ (C_{\tau_p} - \phi C_0 e^{\eta \tau_p}) 1_{\{\phi C_0 e^{\eta \tau_p} < C_{\tau_p} < \phi C_0 e^{\eta \tau_p} + (L e^{\delta \tau_p} - X_{\tau_p})\}},
\]  

(23)

The first term on the right-hand side corresponds to the case where the corporate is able
to cover all the deficits of the pension fund. The second term represents its inability to
do so. After paying back to its own creditors, the corporate can pay the remainder to the
beneficiary.

If \( \tau_p > T \), i.e. the pension fund’s assets have not hit the threshold throughout \([0, T]\), the
sponsoring company needs to provide a guaranteed amount depending on its own funding
situation. \( \tau_p > T \) implies that \( X_T > \eta L e^{\delta T} = \eta L_T \). If, furthermore, \( X_T \geq L_T \), the
pension fund’s assets are sufficient to provide the promised pension payment (note that \( \psi_L(X_T) \geq L_T \)). The sponsoring company does not have to provide the guaranteed amount
in this case. On the other side, if \( \eta L_T < X_T < L_T \), the sponsoring company must cover
the following deficit:

\[
\Phi_c(T) = (L_T - X_T) 1_{\{C_T > \phi C_0 e^{\eta T} + (L_T - X_T)\}} 1_{\{\eta L_T < X_T < L_T\}}
+ (C_T - \phi C_0 e^{\eta T}) 1_{\{\phi C_0 e^{\eta T} < C_T < \phi C_0 e^{\eta T} + (L_T - X_T)\}} 1_{\{\eta L_T < X_T < L_T\}}.
\]  

(24)

Unlike in the case where \( \tau_p \leq T \), we do not know the value of \( X_T \). Again, here the sponsor
provides either a full or a partial guarantee, depending on its own funding situation.

5.2.1. “Premium” of the sponsor. In return for offering downside protection to the pension
fund, the sponsor is entitled to a reward. Under this framework, we therefore also calculate
the “premium-like” payment the sponsor obtains as a compensation for providing support.
We can then compare this premium with the one obtained under the pension guarantee
fund scheme. The premium needs to be determined under the risk-neutral measure \( P^* \).
According to the fair contract principle, i.e. the premium the pension fund pays to the
sponsoring corporate should equal the expected discounted payment the sponsor provides, the premium is determined by

\[ S_{C_0} = E^* [e^{-rT} \Phi_e(\tau_p) 1_{\{\tau_p \leq T\}}] + E^* [e^{-rT} \Phi_e(T) 1_{\{\tau_p > T\}}] \]

\[ = E^* \left[ e^{-rTp} \left( (Le^{\delta \tau_p} - X_{\tau_p}) 1_{\{C_{\tau_p} > \phi C_0 e^{g \eta \tau_p} + (Le^{\delta \tau_p} - X_{\tau_p})\}} + (C_{\tau_p} - \phi C_0 e^{g \eta \tau_p}) 1_{\{\phi C_0 e^{g \eta \tau_p} < C_{\tau_p} < \phi C_0 e^{g \eta \tau_p} + (Le^{\delta \tau_p} - X_{\tau_p})\}) \right) 1_{\{\tau_p \leq T\}} \right] + E^* [e^{-rT} \Phi_e(T) 1_{\{\tau_p > T\}}] \]

\[ = E^* \left[ e^{-rTp} \left( (1 - \eta)Le^{\delta \tau_p} 1_{\{C_{\tau_p} > \phi C_0 e^{g \eta \tau_p} + (1 - \eta)Le^{\delta \tau_p}\}} + (C_{\tau_p} - \phi C_0 e^{g \eta \tau_p}) 1_{\{\phi C_0 e^{g \eta \tau_p} < C_{\tau_p} < \phi C_0 e^{g \eta \tau_p} + (1 - \eta)Le^{\delta \tau_p}\}} \right) 1_{\{\tau_p \leq T\}} \right] + E^* \left[ e^{-rT} \left( (LT - X_T) 1_{\{C_T > \phi C_0 e^{g \tau T} + (LT - X_T)\}} \right) 1_{\{\eta L_T < X_T < L_T\}} \right] + (C_T - \phi C_0 e^{g \tau T}) 1_{\{\phi C_0 e^{g \tau T} < C_T < \phi C_0 e^{g \tau T} + (LT - X_T)\}} 1_{\{\eta L_T < X_T < L_T\}} \right] \]

\[(25)\]

where \( E^* \) denotes the expected value under the risk-neutral measure \( P^* \). From step 2 to 3 we have used the fact that \( X_{\tau_p} = \eta Le^{\delta \tau_p} \).

**Proposition 5.5.** The implicit fair premium that the sponsoring corporate gets from the pension fund can be decomposed into two parts. The first component is

\[ E^* [e^{-rT} \Phi_e(\tau_p) 1_{\{\tau_p \leq T\}}] \]

\[ = \int_0^T e^{-rs} (1 - \eta) Le^{gs} N \left( -\frac{d_1(s)}{\sqrt{s}} \right) \tilde{f}(s) ds \]

\[ + \int_0^T e^{-rs} C_0 \exp \left\{ \left( r - \frac{1}{2} \sigma_c^2 \right) s + \sigma_c \rho \frac{\ln \frac{\eta L_T}{X_0} - (r - \delta - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma} \right\} \cdot \left( \int_{d_1(s)/\sqrt{s}}^{d_2(s)/\sqrt{s}} \exp \left\{ \sigma_c \sqrt{1 - \rho^2} \sqrt{s} x \right\} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right) \tilde{f}(s) ds \]

\[ - \int_0^T e^{-rs} \phi C_0 e^{gs} \left( N \left( \frac{d_1(s)}{\sqrt{s}} \right) - N \left( \frac{d_2(s)}{\sqrt{s}} \right) \right) \tilde{f}(s) ds \]

\[ d_1(s) = \frac{\phi C_0 e^{g(s + (1 - \eta)Le^{gs})} - (r - \frac{1}{2} \sigma_c^2) s - \sigma_c \rho \frac{\ln \frac{\eta L_T}{X_0} - (r - \delta - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma}}{\sigma_c \sqrt{1 - \rho^2}} \]

\[ d_2(s) = \frac{\phi C_0 e^{gs} - (r - \frac{1}{2} \sigma_c^2) s - \sigma_c \rho \frac{\ln \frac{\eta L_T}{X_0} - (r - \delta - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma}}{\sigma_c \sqrt{1 - \rho^2}} \]
where $\tilde{f}(s)$ is the density of the first hitting time $\tau_p$ under $P^*$:

$$
\tilde{f}(s) = -\frac{\ln \frac{\eta L}{X_0}}{\sigma \theta s^{3/2}} n \left( \frac{\ln \frac{\eta L}{X_0} - \tilde{r}s}{\sigma \theta \sqrt{s}} \right).
$$

(26)

And the second component is

$$
E^*[e^{-rT}\Phi_c(T)1_{\{\tau_p>T\}}] = P^*(\tau_p>T) \int_{d_{x1}}^{d_{x2}} \int_{d_{y1}}^{d_{y2}} e^{-rT} \left( L_T - X_0 \exp \left\{ (r - \frac{1}{2}\theta^2\sigma^2)T + \theta \sigma \sqrt{T} x \right\} \right)
$$

$$
\cdot \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} (x^2 + y^2 - 2\rho xy) \right\} \ dy \ dx
$$

$$
+ P^*(\tau_p>T) \int_{d_{x1}}^{d_{x2}} \int_{d_{y1}}^{d_{y2}} e^{-rT} \left( C_0 \exp \{ (r - \frac{1}{2}\sigma_c^2)T + \sigma_c \sqrt{T} y \} - \phi C_0 e^{gT} \right)
$$

$$
\cdot \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} (x^2 + y^2 - 2\rho xy) \right\} \ dy \ dx
$$

$$
d_{x1} = \ln \frac{\eta L \phi}{X_0} - (r - \frac{1}{2}\theta^2\sigma^2)T \ \frac{\theta \sigma \sqrt{T}}{\phi \sigma \sqrt{T}}, \quad d_{x2} = \ln \frac{\eta L \phi}{X_0} - (r - \frac{1}{2}\theta^2\sigma^2)T \ \frac{\theta \sigma \sqrt{T}}{\phi \sigma \sqrt{T}}
$$

$$
d_{y1} = \ln \frac{\phi C_0 e^{gT} + X_0 \exp \{ (r - \frac{1}{2}\theta^2\sigma^2)T + \theta \sigma \sqrt{T} x \} - L_T}{\sigma_c \sqrt{T}} - (r - \frac{1}{2}\sigma_c^2)T \ \frac{\phi C_0 e^{gT}}{c_0} - (r - \frac{1}{2}\sigma_c^2)T \ \frac{\phi C_0 e^{gT}}{c_0}
$$

$$
, \quad d_{y2} = \frac{\phi C_0 e^{gT}}{c_0} - (r - \frac{1}{2}\sigma_c^2)T
$$

PROOF: The proof is found in Appendix 6.3. 

The integrals in the premium calculation can be calculated numerically. Table 6 provides the outcomes for the premiums. Again we make several observations. First, a higher barrier level for the pension fund implies a higher default rate. The premium part which accounts for the premature default increases, whereas the premium part which accounts for the survival of the maturity date decreases. Consequently, the required sponsor premium does not move monotonically in the $\eta$ level. Second, a higher correlation increases the probability of a double default, which implies that should the pension fund become underfunded, the firm will be unable to provide the guarantee. Hence, high correlation leads to a lower sponsor premium. Third, increasing the pension fund’s exposure to risky assets also increases the fair compensation for the sponsor. In this case, the beneficiaries need to give up a larger part of the potential surplus to the sponsor. Fourth, the premium due for a sponsor guarantee is lower than that paid to the PGF. This is due to the fact that
\[ \eta = 0.1 \quad \eta = 0.3 \quad \eta = 0.5 \quad \eta = 0.7 \quad \eta = 0.9 \]

\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( \rho = -0.5 \) & \hline
\( \theta = 0.4 \) & 4.1888 & 4.1899 & 4.3730 & 5.0057 & 3.5442 \hline
\( \theta = 0.5 \) & 5.4879 & 5.5158 & 6.0586 & 6.5987 & 4.1166 \hline
\( \theta = 0.6 \) & 6.7051 & 6.8585 & 7.7314 & 7.9424 & 4.5554 \hline
\hline
\( \rho = 0 \) & \hline
\( \theta = 0.4 \) & 2.2943 & 2.2945 & 2.3583 & 2.7665 & 2.4045 \hline
\( \theta = 0.5 \) & 3.0799 & 3.0872 & 3.3025 & 3.8511 & 2.9459 \hline
\( \theta = 0.6 \) & 3.8429 & 3.8848 & 4.2927 & 4.8610 & 3.3925 \hline
\hline
\( \rho = 0.5 \) & \hline
\( \theta = 0.4 \) & 0.8074 & 0.8074 & 0.8012 & 0.8763 & 1.1516 \hline
\( \theta = 0.5 \) & 1.1618 & 1.1604 & 1.1421 & 1.3648 & 1.5928 \hline
\( \theta = 0.6 \) & 1.5294 & 1.5186 & 1.5061 & 1.9164 & 2.0001 \hline
\end{tabular}

Table 6. Fair premium of the sponsor as a function of the pension fund’s barrier level \( \eta \), allocation to risky assets \( \theta \) and correlation coefficient \( \rho \), with parameters: \( r = 0.05; \sigma_c = 0.33; \phi = 0.9; g = 0.02; X_0 = 100; L = 90; \sigma = 0.20; \delta = 0.05. \) and \( T = 15. \)

The PGF provides a full guarantee, whereas the sponsor only provides a partial guarantee.

The beneficiary’s expected rate of return can be determined by first computing the expected payoff of the contract:

\[
E[\tilde{V}_L] = E[(\psi_L(X_T) + \Phi_c(T))1_{\{\tau_p > T\}}] + E\left[L e^{\delta \tau_p} e^{r(T-\tau_p)} 1_{\{\tau_p < T, C_{\tau_p} > \phi C_0 e^{g \tau_p} + L e^{\delta \tau_p} - X_{\tau_p}\}}\right] \\
+ E\left[(X_{\tau_p} + C_{\tau_p} - \phi C_0 e^{g \tau_p}) e^{r(T-\tau_p)} 1_{\{\tau_p < T, C_{\tau_p} < \phi C_0 e^{g \tau_p} + L e^{\delta \tau_p} - X_{\tau_p}\}}\right] \\
+ E\left[X_{\tau_p} e^{r(T-\tau_p)} 1_{\{\tau_p < T, C_{\tau_p} < \phi C_0 e^{g \tau_p}\}}\right] 
\]

\( E[\psi_L(X_T)1_{\{\tau_p > T\}}] \) has already been determined in the benchmark model. \( E[\Phi_c(T)1_{\{\tau_p > T\}}] \) can be calculated using the similar derivation of the second component of the premium in
Proposition 5.5. It turns out that

\[
E[\Phi_c(T)1_{\{\tau_p > T\}}] = P(\tau_p > T) \int_{\tilde{d}_{x_1}}^{\tilde{d}_{x_2}} \int_{\tilde{d}_{y_1}}^{\infty} \left( L_T - X_0 \exp \left\{ (\tilde{\mu} - \frac{1}{2} \theta^2 \sigma^2)T + \theta \sigma \sqrt{Tx} \right\} \right)
\]

\[
\frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} (x^2 + y^2 - 2\rho xy) \right\} dy \, dx
\]

\[
+ P(\tau_p > T) \int_{\tilde{d}_{x_1}}^{\tilde{d}_{x_2}} \int_{\tilde{d}_{y_1}}^{\tilde{d}_{y_2}} e^{-rT} \left( C_0 \exp \{ (\mu_c - \frac{1}{2} \sigma_c^2)T + \sigma_c \sqrt{Ty} \} - \phi C_0 e^{gT} \right)
\]

\[
\frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} (x^2 + y^2 - 2\rho xy) \right\} dy \, dx
\]

\[\text{(28)}\]

\[
\tilde{d}_{x_1} = \frac{\ln \eta_{L_T} - (\tilde{\mu} - \frac{1}{2} \theta^2 \sigma^2)T}{\theta \sigma \sqrt{T}}, \quad \tilde{d}_{x_2} = \frac{\ln \frac{L_T}{X_0} - (\tilde{\mu} - \frac{1}{2} \theta^2 \sigma^2)T}{\theta \sigma \sqrt{T}}
\]

\[
\tilde{d}_{y_1} = \frac{\ln \frac{\phi C_0 e^{gT} + X_0 \exp \{ (\tilde{\mu} - \frac{1}{2} \theta^2 \sigma^2)T + \theta \sigma \sqrt{Tx} \} - L_T}{C_0}}{\sigma_c \sqrt{T}} - (\mu_c - \frac{1}{2} \sigma_c^2)T,
\]

\[
\tilde{d}_{y_2} = \frac{\ln \frac{\phi C_0 e^{gT} - (\mu_c - \frac{1}{2} \sigma_c^2)T}{C_0}}{\sigma_c \sqrt{T}}.
\]

The last three terms give the expected rebate payments for different scenarios: the second term illustrates the case in which the firm’s assets are sufficient to provide the guaranteed amount; the third term describes the case in which the sponsor can partially provide the guaranteed amount and the final term represents the case in which the sponsor is already in default. The detailed calculation of these three terms is given in the following proposition.
Proposition 5.6. The three components of the expected rebate of the beneficiary are given by

\[
E\left[ Le^{\delta r_p} e^{r(T - \tau_p)} 1\{C_{\tau_p} \geq \phi C_0 e^{\theta r_p} + (Le^{\delta r_p} - X_{\tau_p})\} \right] = L \int_0^T e^{\delta s} e^{r(T - s)} N\left( \frac{-\bar{d}_1(s)}{\sqrt{s}} \right) \bar{f}(s) ds; \\
E\left[ (X_{\tau_p} + C_{\tau_p} - \phi C_0 e^{\theta r_p}) e^{r(T - \tau_p)} 1\{\phi C_0 e^{\theta r_p} < C_{\tau_p} < \phi C_0 e^{\theta r_p} + (Le^{\delta r_p} - X_{\tau_p})\} \right] = \int_0^T (\eta L e^{\delta s} - \phi C_0 e^{\theta r_p}) e^{r(T - s)} \left( N\left( \frac{\bar{d}_1(s)}{\sqrt{s}} \right) - N\left( \frac{\bar{d}_2(s)}{\sqrt{s}} \right) \right) \bar{f}(s) ds + \int_0^T C_0 e^{r(T - s)} \exp \left\{ \left( \mu_c - \frac{1}{2} \sigma_c^2 \right) s + \sigma_c \rho \frac{\ln \frac{\eta L}{T_0} - (\bar{\mu} - \bar{\delta} - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma} \right\} \cdot \left( \int_{\bar{d}_1(s)/\sqrt{s}}^{\bar{d}_2(s)/\sqrt{s}} \exp \left\{ \sigma_c \sqrt{1 - \rho^2} \sqrt{s} x \right\} \frac{1}{\sqrt{2\pi \sigma_c^2}} e^{-x^2/2} dx \right) \bar{f}(s) ds
\]

\[
E\left[ X_{\tau_p} e^{r(T - \tau_p)} 1\{C_{\tau_p} \leq \phi C_0 e^{\theta r_p}\} \right] = \eta L \int_0^T e^{\delta s} e^{r(T - s)} N\left( \frac{\bar{d}_2(s)}{\sqrt{s}} \right) \bar{f}(s) ds; \\
\bar{d}_1(s) = \frac{\ln \frac{\phi C_0 e^{\theta s} + (1 - \eta) Le^{\delta s}}{C_0} - (\mu_c - \frac{1}{2} \sigma_c^2) s - \sigma_c \rho \frac{\ln \frac{\eta L}{T_0} - (\bar{\mu} - \bar{\delta} - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma}}{\sigma_c \sqrt{1 - \rho^2}} \\
\bar{d}_2(s) = \frac{\ln \frac{\phi C_0 e^{\theta s}}{C_0} - (\mu_c - \frac{1}{2} \sigma_c^2) s - \sigma_c \rho \frac{\ln \frac{\eta L}{T_0} - (\bar{\mu} - \bar{\delta} - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma}}{\sigma_c \sqrt{1 - \rho^2}} \\
\bar{f}(s) = -\frac{\ln \frac{\eta L}{T_0}}{\sigma \theta s^{3/2}} \left( \frac{\ln \frac{\eta L}{T_0} - \bar{\mu} s}{\sigma \theta \sqrt{s}} \right).
\]

Proof: The proof is very similar to that of Proposition 5.5.

Table 7 provides some values of the expected log return in case of sponsor support and allows to observe the following. The effects of \( \beta \) and \( \theta \) are similar as in the case of solvency requirements. This is due to the fact that default/intervention is triggered by the same event, i.e. by the underfunding of the pension fund. Second, the expected rate of return can be higher or lower than under regulatory own funds. For very conservative investment strategies (here \( \theta = 0.1 \)), sponsor support leads to a lower expected log-return. The “sponsor premium” is relatively expensive for such a low \( \theta \). For more aggressive investment strategies, the beneficiary benefits from the sponsor support. The higher the holding
$\beta = 0.6$, $\beta = 0.7$, $\beta = 0.8$, $\beta = 0.9$.

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<tr>
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<td>0.493</td>
<td>0.506</td>
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<td>0.468</td>
<td>0.487</td>
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<tr>
<td>$\theta$</td>
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<td>0.420</td>
<td>0.444</td>
<td>0.467</td>
<td>0.490</td>
</tr>
</tbody>
</table>

**Table 7.** The expected rate of return with sponsor support as a function of the participation rate $\beta$ and allocation to risky assets $\theta$ with parameters:

$T = 15$, $r = 0.05$, $\mu_c = 0.1$, $\sigma_c = 0.33$, $\mu = 0.08$, $\sigma = 0.2$, $\theta = 0.5$, $X_0 = 100$, $L = 90$, $\phi = 0.8$, $\eta = 0.9$, $\rho = 0.5$ and $\delta = 0.05$.

of the risky asset, the less the beneficiary can profit from the sponsor support. Hence, unlike the support by a PGF, sponsor guarantee does not provide strong protection to the beneficiary, particularly if the pension fund pursues very aggressive investment strategies.

Compared to the benchmark model, this scheme provides better guarantees to the beneficiaries. Unlike the pension guarantee fund case, there is a chance that the beneficiaries may not obtain the full guarantee amount in spite of the guarantee provided by the sponsor. That depends on the financial position of the sponsor and the debt level.\(^{15}\) Where both the pension fund and the sponsor default, the pension fund’s deficit (to the beneficiary) cannot be covered at all. When the pension fund defaults, whereas the sponsor is still solvent, the pension fund’s deficit might be only partially covered if the sponsor’s asset only just exceeds its debt level.\(^{16}\) The shortfall probability in case of sponsor guarantee is stated in the following proposition.

\(^{15}\)This sponsor vulnerability is analyzed further in Broeders (2010).

\(^{16}\)Note that we disregard any feedback between the pension fund and the sponsor.
**Proposition 5.7.** The probability that the pension’s fund’s deficit cannot be covered fully can be decomposed into the following two probabilities:

\[
E[1_{\{X_T \leq L_T\}}1_{\{\tau_p > T\}}] + E[1_{\{X_T \leq L_T\}}1_{\{\tau_p \leq T\}}]
\]

\[= P(\tau_p > T) - P(X_T > L_T, \tau_p > T)\]

\[+ P(C_{\tau_p} < \phi C_0 e^{\gamma \tau_p}) + P(C_{\tau_p} < \phi C_0 e^{\gamma \tau_p} + (L e^{\delta \tau_p} - X_{\tau_p}))\]

The first term is given by:

\[P(\tau_p > T) - P(X_T > L_T, \tau_p > T)\]

\[= P(\tau_p > T) - N\left(d^-((B_0)^2, X_0, Y e^{-\mu T})\right) + \left(\frac{B_0}{X_0}\right)^{2(\tilde{\mu} - \delta - \frac{\theta^2}{2} \sigma^2)} N\left(d^-((B_0)^2, X_0, Y e^{-\tilde{\mu} T})\right)\]

The second term is given by

\[P(C_{\tau_p} < \phi C_0 e^{\gamma \tau_p}) + P(C_{\tau_p} < \phi C_0 e^{\gamma \tau_p} + (L e^{\delta \tau_p} - X_{\tau_p}))\]

\[= E\left[1\{C_{\tau_p} < \phi C_0 e^{\gamma \tau_p} + (L e^{\delta \tau_p} - X_{\tau_p})\}\right] = \int_0^T N\left(\frac{\bar{d}_1(s)}{\sqrt{s}}\right) \tilde{f}(s) ds\]

where \(\tilde{\mu} = \tilde{\mu} - \delta - \frac{\theta^2}{2} \sigma^2\) and \(\tilde{f}(s)\) is the density of \(\tau_p\) under the market measure \(P\). \(\bar{d}_1\) and \(\tilde{f}(s)\) are the same as in Proposition 5.6.

**Proof:** The proof is very similar as Proposition 5.5.

Table 8 illustrates several shortfall probabilities under a sponsor guarantee scheme. The shortfall probabilities lie between those of the other two schemes. Due to the partial guarantee provided by the sponsor the shortfall probability is much lower than under the case of solvency requirements, but naturally far higher than in the PGF case.

**6. Conclusion**

Pension benefits can be secured in various ways. This paper compares solvency requirements to a pension guarantee fund and sponsor support. Solvency requirements provide an internal guarantee, while a pension guarantee fund and sponsor support act as external guarantee mechanisms. Comparisons are carried out by calculating the beneficiary’s expected log-return and the shortfall probability where the beneficiary does not obtain the promised pension payment in full.
The following key conclusions follow from our analysis.

- We show that it is possible to compare different security mechanisms aimed at protecting pension benefits, using appropriate tools from finance. We focus on comparing expected log-return and shortfall probabilities.
- Contingent claims analysis can be used to determine fair compensation rates for a pension guarantee fund and a sponsor in exchange for offering downside protection to a pension fund and its beneficiaries. One of the key parameters in the pricing mechanism is the correlation between the return on the pension fund’s assets and the sponsor’s assets.
- We see that a PGF provides a high level of protection to the beneficiary, particularly if the pension funds pursues an aggressive investment policy, whereas solvency requirements provide only relatively weak protection. However, in specific cases solvency requirements lead to a higher expected return as the beneficiaries need not pay an insurance premium.
- Compared to a system based on solvency requirements alone, support by a pension guarantee fund or by the sponsor offers better downside protection for those pension funds with an aggressive investment policy, whereas it leaves beneficiaries worse off if the pension funds have a conservative investment policy.
- Compared to the protection provided by a PGF, sponsor support provides only a partial guarantee of full pension payment. Consequently, this protection comes at

<table>
<thead>
<tr>
<th>$\eta$ = 0.6</th>
<th>$\eta$ = 0.7</th>
<th>$\eta$ = 0.8</th>
<th>$\eta$ = 0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ = 0.1</td>
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<td>0.0286</td>
<td>0.0286</td>
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<tr>
<td>$\theta$ = 0.3</td>
<td>0.1792</td>
<td>0.1780</td>
<td>0.1742</td>
</tr>
<tr>
<td>$\theta$ = 0.5</td>
<td>0.2506</td>
<td>0.2480</td>
<td>0.2637</td>
</tr>
<tr>
<td>$\theta$ = 0.7</td>
<td>0.2993</td>
<td>0.3113</td>
<td>0.3417</td>
</tr>
<tr>
<td>$\theta$ = 0.9</td>
<td>0.3496</td>
<td>0.3729</td>
<td>0.4006</td>
</tr>
</tbody>
</table>

Table 8. Shortfall probability under the solvency regime as a function of barrier level $\eta$ and allocation to risky assets $\theta$ with parameters: $\mu = 0.08$, $\sigma = 0.2$, $T = 15$, $X_0 = 100$, $L = 90$, $\delta = r = 0.05$. 
an additional cost which is, however, below the premium resulting from the PGF insurance.

These conclusions are drawn from a stylized setting. Obviously, both the model and the analysis can be extended in various ways. As we deal with nonlinear payoff structures, it is desirable to also include higher moments in a true comparison of different security mechanisms. We leave the calculations of higher moments to future research. Other areas of future extensions are Loss Given Default as a measure of risk and a utility analysis to incorporate beneficiaries' preferences.
Table 9. Calibration table.

### Appendix

6.1. **Appendix A.** Table 9 presents the calibration used throughout the paper.

6.2. **Appendix B: Derivation of the fair premium PGF receives.** The term $E^*[e^{-rT}\max(L\delta T - X_T, 0)1_{\{\tau_c > T\}}]$ is the price of a rainbow down-and-out put option maturing at date $T$. The pricing formulas for rainbow barrier knock–out call or put options can be expressed in terms of the cumulative bivariate normal distribution function with correlation coefficient $\rho$ and a detailed derivation can be found in, e.g., Carr (1996) or Haug (2007). The pricing formulas for the rainbow barrier knock–out call or put options
can be expressed in terms of the cumulative bivariate normal distribution function with correlation coefficient \( \rho \) as follows (see Haug (2007)):

\[
RBO = \kappa X_0 \left[ M(\kappa d_1, \nu e_1; -\kappa \nu \rho) - e^{-\frac{2(r - g - \frac{1}{2} \sigma^2)}{\sigma^2} \ln(\phi C_0 / C_0)} M(\kappa d_3, \nu e_3; -\kappa \nu \rho) \right]
- \kappa Le^{\delta T} e^{-rT} \left[ M(\kappa d_2, \nu e_2; -\kappa \nu \rho) - e^{-\frac{2(r - g - \frac{1}{2} \sigma^2)}{\sigma^2} \ln(\phi C_0 / C_0)} M(\kappa d_4, \nu e_4; -\kappa \nu \rho) \right]
\]

where

\[
\begin{align*}
d_1 &= \frac{\ln(X_0/(Le^{\delta T})) + (r + \frac{1}{2} \theta^2 \sigma^2) T}{\theta \sigma \sqrt{T}} \\
d_2 &= d_1 - \sigma \theta \sqrt{T} \\
d_3 &= d_1 + \frac{2 \rho \ln(\phi C_0 / C_0)}{\sigma_c \sqrt{T}} \\
d_4 &= d_2 + \frac{2 \rho \ln(\phi C_0 / C_0)}{\sigma_c \sqrt{T}} \\
e_1 &= \frac{\ln(\phi C_0 / C_0) - (r - g - \frac{1}{2} \sigma^2 + \rho \sigma \theta \sigma_c) T}{\sigma_c \sqrt{T}} \\
e_2 &= e_1 + \rho \sigma \theta \sqrt{T} \\
e_3 &= e_1 - \frac{2 \ln(\phi C_0 / C_0)}{\sigma_c \sqrt{T}} \\
e_4 &= e_2 - \frac{2 \ln(\phi C_0 / C_0)}{\sigma_c \sqrt{T}}
\end{align*}
\]

with \( M(a, b; \rho) \) denoting the cumulative bivariate normal distribution function defined as

\[
M(a, b; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp \left\{ -\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right\} dx dy
\]

and where

- \( \kappa = 1 \) and \( \nu = -1 \) for a down–and–out call,
- \( \kappa = 1 \) and \( \nu = 1 \) for an up–and–out call,
- \( \kappa = -1 \) and \( \nu = -1 \) for a down–and–out put,
- \( \kappa = -1 \) and \( \nu = 1 \) for an up–and–out put.

The above formulae are already adjusted to an exponentially time-increasing barrier (through the \(-g\) term).

In order to compute \( E^* [e^{-r \tau_c} \max(Le^{\delta \tau_c} - X_{\tau_c}, 0)1_{\{\tau_c \leq T\}}] \), we need to know the distribution of \( \tau_c \). Under \( P^* \), the intervention probability is given by

\[
P^*(\tau_c \leq s) = N \left( \frac{\ln(\phi) - \hat{r}_c s}{\sigma_c \sqrt{s}} \right) + \left( 1 - N \left( \frac{\ln(\phi) + \hat{r}_c s}{\sigma_c \sqrt{s}} \right) \right)
\]

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with \( \hat{r}_c = r - g - \frac{\sigma^2}{2} \). The density is then given by taking the first derivative with respect to \( s \). Using the fact that

\[
 n \left( \frac{\ln \phi + \hat{r}_c s}{\sigma_c \sqrt{s}} \right) = \left( \frac{1}{\phi} \right)^{\frac{2\hat{r}_c}{\sigma_c^2}} n \left( \frac{\ln \phi - \hat{r}_c s}{\sigma_c \sqrt{s}} \right)
\]

with \( n(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \), we obtain the density:

\[
 f(s) = \frac{\partial P^*(\tau_c \leq s)}{\partial s} = -\ln \phi \sigma_c s^{3/2} n \left( \frac{\ln \phi - \hat{r}_c s}{\sigma_c \sqrt{s}} \right).
\]

To further calculate the premium, note first that at time \( \tau_c \), we have assumed that the pension fund terminates its operations and is taken over by the pension guarantee fund. At time \( \tau_c \), it holds that

\[
 C_{\tau_c} = \phi C_0 e^{g\tau_c} \Rightarrow \tilde{W}^1_{\tau_c} = \rho W^1_{\tau_c} + \sqrt{1 - \rho^2} W^2_{\tau_c} = \frac{1}{\sigma_c} \left( \ln \phi - \left( r - g - \frac{1}{2} \sigma^2 \right) \tau_c \right).
\]

Since \( W^1_{\tau_c} \) and \( \tilde{W}^1_{\tau_c} \) are correlated with a coefficient \( \rho \), it implies we can rewrite \( X_{\tau_c} \) as follows:

\[
 X_{\tau_c} = X_0 \exp \left\{ \left( r - \frac{1}{2} \theta^2 \sigma^2 \right) \tau_c + \theta \sigma \left( \rho \tilde{W}^1_{\tau_c} + \sqrt{1 - \rho^2} \tilde{W}^2_{\tau_c} \right) \right\} = X_0 \exp \left\{ \left( r - \frac{1}{2} \theta^2 \sigma^2 \right) \tau_c + \theta \sigma \left( \frac{\rho}{\sigma_c} \left( \ln \phi - \left( r - g - \frac{1}{2} \sigma^2 \right) \tau_c \right) + \sqrt{1 - \rho^2} \tilde{W}^2_{\tau_c} \right) \right\}
\]

where \( \tilde{W}^2_{\tau_c} \) is independent of \( \tilde{W}^1_{\tau_c} \) and is normally distributed random with mean 0 and variance \( \tau_c \) given \( \tau_c \). Given \( \tau_c \), the condition \( X_{\tau_c} < Le^{\delta \tau_c} \) can be transformed to

\[
 \tilde{W}^2_{\tau_c} < \frac{\ln \frac{Le^{\delta \tau_c}}{X_0} - \frac{\theta \sigma \rho}{\sigma_c} \ln \phi - \left( r - \frac{1}{2} \theta^2 \sigma^2 - \frac{\theta \sigma \rho}{\sigma_c} \left( r - g - \frac{1}{2} \sigma^2 \right) \right) \tau_c}{\theta \sigma \sqrt{1 - \rho^2}} := d(\tau_c).
\]
Knowing the density of $\tau_c$, we obtain the premium as follows:

$$
E^*[e^{-r\tau_c} \max(Le^{\delta\tau_c} - X_{\tau_c}, 0)1_{\{\tau_c \leq T\}}] = E^*[E[e^{-r\tau_c} (Le^{\delta\tau_c} - X_{\tau_c})1_{\{X_{\tau_c} \leq Le^{\delta\tau_c}\}}]|\tau_c] = E^*[e^{-r\tau_c} E^*[\left(Le^{\delta\tau_c} - X_0 \exp\left\{\left(r - \frac{1}{2} \theta^2 \sigma^2\right) \tau_c + \theta \sigma \left(\ln \phi - \left(r - g - \frac{1}{2} \sigma^2_c\right) \tau_c\right) + \sqrt{1 - \rho^2 W_{\tau_c}^2}\right)\right]1_{\{\tilde{W}_{\tau_c} < d(\tau_c)\}}]|\tau_c]
$$

$$
= \int_0^T e^{-rs} \int_{-\infty}^{d(s)/\sqrt{s}} \left(Le^{\delta s} - X_0 \exp\left\{\left(r - \frac{1}{2} \theta^2 \sigma^2\right)s + \theta \sigma \left(\ln \phi - \left(r - g - \frac{1}{2} \sigma^2_c\right)s + \sqrt{1 - \rho^2 \sqrt{s}} x\right)\right)\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx f(s) ds.
$$

### 6.3. Derivation of fair premium in case of sponsor support.

The premium can be decomposed into two parts: $E^*[e^{-r\tau_p} \Phi_c(\tau_p)1_{\{\tau_p \leq T\}}]$ and $E^*[e^{-rT} \Phi_c(T)1_{\{\tau_p > T\}}]$. In the former expectation in which $\tau_p \leq T$, note that at time $\tau_p$, $X_{\tau_p} = \eta Le^{\delta\tau_p}$ and $C_{\tau_p}$ can be rewritten as

$$
C_{\tau_p} = C_0 \exp\left\{\left(r - \frac{1}{2} \sigma^2_c\right) \tau_p + \sigma_c \frac{\ln \eta L - (r - \delta - \frac{1}{2} \theta^2 \sigma^2) s}{\theta \sigma} + \sqrt{1 - \rho^2 \sigma_c \sqrt{\tau_p} W_{\tau_p}^2}\right\}.
$$

Upon the reformulation and following similar calculations as in Section 4, we can determine the premium, once we know the density of the first hitting time $\tau_p$ under $P^*$. Under $P^*$, the intervention probability is given by

$$
P^*(\tau_p \leq s) = N\left(\frac{\ln(\eta L) - \hat{r}T}{\sigma \theta \sqrt{T}}\right) + \left(\frac{X_0}{\eta L}\right)^{\frac{-\hat{r}}{\sigma^2}} N\left(\frac{\ln(\eta L) + \hat{r}T}{\sigma \theta \sqrt{T}}\right)
$$

with $\hat{r} = r - \delta - \frac{\theta^2 \sigma^2}{2}$. The density is then given by taking the first derivative with respect to $s$ and can be expressed as in (26). It is then straightforward to write down the expected value.
The second expectation can be derived as follows:

\[
E^*[e^{-rT} \Phi_c(T)1_{\{\tau_p > T\}}]
\]

\[
= E^*[E^*[e^{-rT} \Phi_c(T)1_{\{\tau_p > T\}} | \tau_p > T]]
\]

\[
= E^*[\{\tau_p > T\} E^*[e^{-rT} (L_T - X_T) 1_{\{C_T > \phi C_0 e^{\sigma T} + (L_T - X_T)\}} 1_{\{\eta L_T < X_T < L_T\}}
\]

\[
+ (C_T - \phi C_0 e^{\sigma T}) 1_{\{\phi C_0 e^{\sigma T} + (L_T - X_T)\}} 1_{\{\eta L_T < X_T < L_T\}\}| \tau_p > T]]
\]

\[
= P^*(\tau_p > T) \int_{d_{x_1}}^{d_{x_2}} \int_{d_{y_1}}^{\infty} e^{-rT} \left( L_T - X_0 \exp \left\{ \left( r - \frac{1}{2} \theta^2 \sigma^2 \right) T + \theta \sigma \sqrt{T} x \right\} \right)
\]

\[
\frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left( x^2 + y^2 - 2\rho xy \right) \right\} dy dx
\]

\[
+ P^*(\tau_p > T) \int_{d_{x_1}}^{d_{x_2}} \int_{d_{y_2}}^{d_{y_1}} e^{-rT} \left( C_0 \exp \left\{ (r - \frac{1}{2} \sigma_c^2) T + \sigma_c \sqrt{T} \right\} - \phi C_0 e^{\sigma T} \right)
\]

\[
\frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left( x^2 + y^2 - 2\rho xy \right) \right\} dy dx
\]

The first equality results from the property of the iterated expectations. The second equality is valid because \{\tau_p > T\} implies that the pension fund’s asset \( X_T \) is above the barrier level \( \eta L_T \). In the third equality, we use the fact that \( \ln X \) and \( \ln C \) follow the cumulative bivariate normal distribution with correlation coefficient \( \rho \). \( d_{x_1} \) and \( d_{x_2} \) are obtained by expressing \( X_T \) as follows:

\[
X_T = X_0 \exp \left\{ (r - \frac{1}{2} \sigma^2 \theta^2) T + \sigma \theta \sqrt{T} x \right\}
\]

where \( x \) is a standard normally distributed random variable and letting \( \eta L_T < X_T < L_T \). Similarly \( d_{y_1} \) and \( d_{y_2} \) are obtained by expressing \( C_T \) as follows:

\[
C_T = C_0 \exp \left\{ (r - \frac{1}{2} \sigma_c^2) T + \sigma_c \sqrt{T} y \right\}
\]

where \( y \) is again a standard normally distributed random variable and letting \( \phi C_0 e^{\sigma T} < C_T < \phi C_0 e^{\sigma T} + X_T - L_T \). We obtain

\[
d_{x_1} = \frac{\ln \frac{X_T}{X_0} - (r - \frac{1}{2} \theta^2 \sigma^2) T}{\theta \sigma \sqrt{T}}, \quad d_{x_2} = \frac{\ln \frac{L_T}{X_0} - (r - \frac{1}{2} \theta^2 \sigma^2) T}{\theta \sigma \sqrt{T}}
\]

\[
d_{y_1} = \frac{\ln \frac{\phi C_0 e^{\sigma T} + X_0 \exp \left\{ (r - \frac{1}{2} \theta^2 \sigma^2) T + \theta \sigma \sqrt{T} x \right\} - L_T}{C_0} - (r - \frac{1}{2} \sigma_c^2) T}{\sigma_c \sqrt{T}}, \quad d_{y_2} = \frac{\ln \frac{\phi C_0 e^{\sigma T}}{C_0} - (r - \frac{1}{2} \sigma_c^2) T}{\sigma_c \sqrt{T}}.
\]
The survival probability can be determined by using the reflection principle of the Brownian motions and has already been derived in literature, see e.g. in Haug (2007)

\[ P^*(\tau_p > T) = 1 - N\left(\frac{\ln\left(\frac{nL_0}{X_0}\right) - \left(r - \frac{1}{2} \theta^2 \sigma^2\right)T}{\sigma\theta\sqrt{T}}\right) \]

\[ - \left(\frac{X_0}{\eta L_0}\right)^{-\frac{2\left(r - \frac{1}{2} \theta^2 \sigma^2\right)}{\sigma^2 \theta^2}} N\left(\frac{\ln\left(\frac{nL_0}{X_0}\right) + \left(r - \frac{1}{2} \theta^2 \sigma^2\right)T}{\sigma\theta\sqrt{T}}\right). \quad (32) \]


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