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* Views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.

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Abstract

We build a novel macro-finance model that combines a semi-structural macroeconomic module with arbitrage-free yield-curve dynamics. We estimate it for the United States and the euro area using a Bayesian approach and jointly infer the real equilibrium interest rate (r^*), trend inflation (π^*), and term premia. Similar to Bauer and Rudebusch (2020, AER), π^* and r^* constitute a time-varying trend for the nominal short-term rate in our model, rendering estimated term premia more stable than standard yield curve models operating with time-invariant means. In line with the literature, our r^* estimates display a distinct decline over the last four decades.

Keywords: Natural rate of interest, r^*, equilibrium real rate, arbitrage-free Nelson-Siegel term structure model, term premia, unobserved components, Bayesian estimation
JEL Classification: C11, C32, E43, G12, E44, E52.

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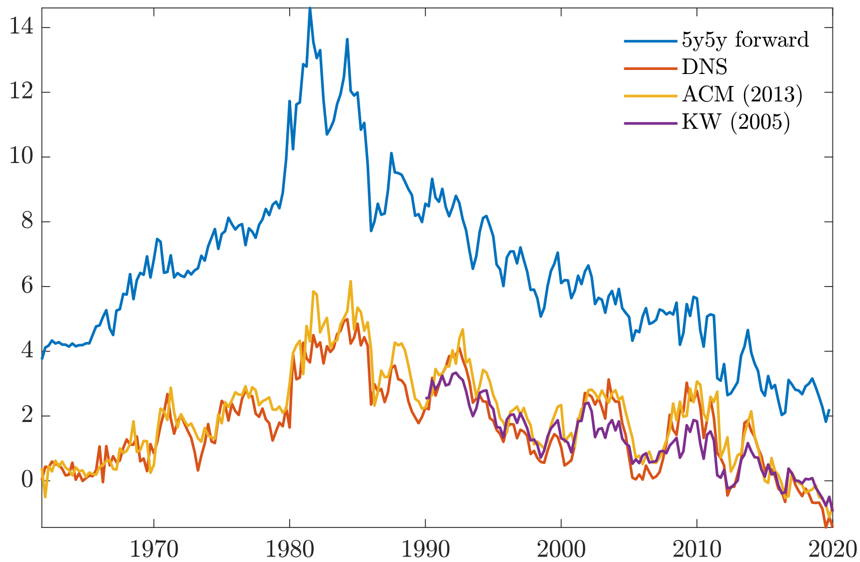
1 Introduction

Since the 1980s short- and long-term bond yields in advanced economies have displayed a protracted downward trend and slumped further in the wake of the global financial crisis. This development is typically attributed to a decline in trend inflation (π^*) and in the natural or equilibrium real rate of interest (r^*) – the latter commonly defined as the real rate consistent with the economy operating at its potential level (in the absence of transitory shocks) or its natural level (in the absence of nominal frictions).[1]

The empirical finance literature studying yield curve dynamics has by and large ignored low-frequency macroeconomic trends relevant for equilibrium yields. Commonly used term structure approaches specify short-rate dynamics as being stationary around a constant mean.[2]

Ignoring such trends has consequences for decompositions of long-term interest rates into average short-rate expectations and term premia: the underestimation of short-rate persistence induces models to attribute the trend in observed bond yields largely to a rise and fall in term premia. Figure 1 illustrates this pattern.

Figure 1: 5-year 5-year forward rates of interest and common term premia estimates for the United States



Note: The figure shows the 5-year, 5-year forward zero coupon bond yield in blue, together with term premium estimates derived from a Dynamic Nelson-Siegel model (DNS) and arbitrage-free term structure models following Adrian et al. (2013) (all authors' calculations) and Kim and Wright (2005), which are taken from FRED.

In a recent paper, Bauer and Rudebusch (2020) address this shortcoming of mean rever-

[1]Inspired by Wicksell (1898), Woodford (2003) established its central role in today's widely used New-Keynesian modeling framework.

[2]Finance models, including those that rely on yield curve information (Dai and Singleton, 2000; Cochrane and Piazzesi, 2005; Diebold and Li, 2006; Adrian et al., 2013) and those incorporating macroeconomic variables (Ang and Piazzesi, 2003; Gürkaynak and Wright, 2012; Wright, 2011; Crump et al., 2018), but also structural macro models, such as Kliem and Meyer-Gohde (2017) and references therein, typically do not take trends in equilibrium rates into account.

sion by constructing a no-arbitrage term-structure model that incorporates a stochastic trend serving as a time-varying attractor for the short-term nominal interest rate. They capture the time-varying trend (i_t^*) in the nominal short-term interest rate using two approaches: either the sum of a survey-based proxy of trend inflation and an average of various (off-model) estimates of the natural real rate ("observed shifting endpoints (OSE)" version of their model); or, they estimate the trend purely based on yield curve information and (relatively tight) Bayesian priors helping the identification of this key latent variable ("estimated shifting endpoints (ESE)" version). Their work importantly expands a sparse earlier literature taking initial steps towards incorporating 'shifting end points' for short-term rate trajectories including Kozicki and Tinsley (2001); Dewachter et al. (2014); Cieslak and Povala (2015); Christensen and Rudebusch (2019); Ajevskis (2020). By incorporating explicitly a trend in the nominal short-term rate the model of Bauer and Rudebusch (2020) attributes a larger share of trends in bond yields to average rate expectations as opposed to the term premium.

Our paper takes the approach of Bauer and Rudebusch (2020) further by combining no-arbitrage yield curve dynamics with a semi-structural macro model akin to Laubach and Williams (2003). As a result the trend i_t^* (the sum of r_t^* and the low-frequency component of inflation π_t^*) is a key driver of both the level of the yield curve and of the state of the macroeconomy.

Specifically, our term structure module is an arbitrage-free affine Nelson-Siegel (AFNS) model with the level factor incorporating a stochastic trend determined by the equilibrium nominal short-term rate i_t^*. The slope and curvature factors, by contrast, are mean-reverting. While trend inflation is specified as a simple random walk, the natural real rate is linked to the expected growth rate of potential output as well as a non-growth component capturing other determinants of r^*. The gap between the actual (model-consistent) real rate and the natural real rate drives the output gap in the IS equation, and the Phillips curve equation links the output gap and inflation.

We estimate the model using a Bayesian approach using quarterly data from 1961Q2 to 2019Q4 for the United States and from 1995Q1 to 2019Q4 for the euro area. We thereby end the sample shortly before the pandemic crisis hit advanced economies: adapting the linear modelling structure to discount highly volatile observations during the pandemic would have taken us too far afield.

We also use survey information on long-run inflation expectations to help pin down trend inflation π^*, as well as one-year ahead expectations of the nominal short rate to inform our econometric model about the speed at which the short rate convergences to its time-varying attractor i^*. For the euro area, we additionally include long-horizon expectations (i.e. 7-10 years ahead) of long-term yields to further inform our estimates of the expectations component in yields and the natural rate.

We find that by accounting for trends in equilibrium rates, term premia exhibit more cyclical behavior, rather than a distinct trend decline as implied by term structure models with fixed long-run means. This result is similar to Bauer and Rudebusch (2020) who report (for the US) that their "term premium estimates exhibit only a modestly decreasing trend and more pronounced cyclical swings". We illustrate that the degree of mean reversion in term premia is a bit more pronounced in our case than in their paper due to our term premium (and slope) being stationary and not loading on the stochastic trend, while in their case it is driven by i^*.

Our r^* estimates for the US and the euro area show a distinct decline from the end-1990s to the end of the sample, a pattern that is shared by Holston et al. (2017) and several other studies in the literature.[3]

Similar to most studies in the literature, r^* is estimated with a sizeable degree of uncertainty. However, while not directly comparable (especially due to using a Bayesian versus multi-stage frequentist approach) our uncertainty bands appear to be narrower than those reported by Holston et al. (2017).

During the 1970s and in the wake of the global financial crisis, our point estimates of r^* are measurably below those from Holston et al. (2017), not unlike estimates in other important econometric papers (see Del Negro et al., 2017, 2019; Fiorentini et al., 2018). Our point estimates are also closer to the average of the model estimates that Bauer and Rudebusch (2020) use for feeding the "OSE" version of their model.[4]

Our proposed macro-finance model responds to the "*... need for further integration of financial and macroeconomic approaches to understanding trends in interest rates*", as recently called for by Kiley (2020). In particular, our paper addresses both the empirical macro literature trying to infer the trajectory of the natural real rate and the empirical finance literature studying yield curve dynamics and its drivers.

The sizeable and expanding literature on estimating the natural real rate[5] reflects the promi-

[3]The literature broadly agrees on a general downward trend in r^* and its fall to levels around zero in the wake of the financial crisis (as far as advanced economies are concerned). It is generally seen as caused by factors including lower productivity and potential output growth, a rise in risk aversion, declining growth rates in the working-age population, rising savings in anticipation of longer retirement periods (at global level), safe-asset scarcity, and possibly increasing inequality and firm profits. See e.g. Gomme et al. (2011); Rachel and Smith (2015); Caballero et al. (2017); Bielecki et al. (2018); Marx et al. (2017); Rannenberg (2018); Gourinchas and Rey (2019); Papetti (2019); Rachel and Summers (2019); Mian et al. (2020) amongst a wide range of studies.

[4]See Panel B of Figure 2 in their paper.

[5]Econometric approaches typically focus on backing out low-frequency components in yields from macroeconomic times series, as e.g. in Laubach and Williams (2003, 2016); Del Negro et al. (2017, 2019); Fiorentini et al. (2018). Structural estimates yielding a contemporaneous stabilization of output gaps from DSGE models have been provided by Edge et al. (2008); Barsky et al. (2014); Cúrdia et al. (2015), and Neri and Gerali (2019), just to name a few. For a review of estimates, drivers and stabilizing properties (for the euro area and the United

nence of r^* in modern monetary macroeconomics and is further stimulated by the present exceptional macroeconomic situation, with policy and short-term money-market interest rates persistently constrained by their effective lower bound. In particular, r^* plays a key role for monetary policy as the natural real rate affects the monetary policy stance:[6] when the actual real rate exceeds its natural counterpart, the resulting positive real rate gap has a contractionary effect on the business cycle and in turn dampens inflation – and vice versa for a negative rate gap.

Notwithstanding the prominence of r^* for determining growth and inflation, the high degree of model uncertainty and measurement issues has prompted some academic economists and policymakers to characterize r^* as a poor guide for policy, as exemplified in the statement by former FOMC member Kevin Warsh asserting "*r-star is not a beacon in the sky but a chimera in the eye*".

Yet, trends and time variation in the natural real rate, even if difficult to observe, are a reality and ignoring them gives rise to misleading model-based results in economics and finance. As a case in point, slow-moving changes in the natural real rate together with potential shifts in the long-run inflation outlook determine the level towards which nominal short-term interest rates are expected to converge in the long run: disregarding such time-variable trends and instead imposing time-invariant equilibria would bias the relative importance attributed to term premia relative to interest rate expectations as determinants of long-term interest rates.

Our modelling approach allows to not only infer the natural short rate, but the whole natural yield curve. We are the first – to the best of our knowledge – to estimate jointly a semi-structural macroeconomic system and an arbitrage-free affine term structure model. By contrast, Brzoza-Brzezina and Kotłowski (2014); Imakubo et al. (2018); Kopp and Williams (2018); Dufrénot et al. (2019) all follow a multi-step approach in which yield curve factors are treated as observables. Moreover, Brzoza-Brzezina and Kotłowski (2014); Imakubo et al. (2018) and Dufrénot et al. (2019) do not provide term-premia estimates. The paper most closely related to our work is Kopp and Williams (2018), yet their approach differs in several aspects. Firstly, the authors choose a model specification in which they replace output and its gap measure with unemployment. Consequently, the real rate trend is not linked to potential output growth, as in Laubach and Williams (2003), but instead follows a simple random walk. Secondly, crucial macroeconomic trends, such as the natural rate of unemployment are treated as observables (subject to measurement error) instead of extracting them from the data. Thirdly, our term structure rules

States), see Brand et al. (2018).

[6] See, e.g., Weber et al. (2008) for a conceptual discussion regarding the usefulness of r^* for monetary policy, and Neiss and Nelson (2003) for a model-based evaluation of the natural rate gap as policy stance indicator.

out riskless arbitrage across bond prices. Finally, we present estimation results for both the United States and the euro area.

The main body of the paper is organized as follows. Section 2 describes the macro-finance term structure model and compares it to the Bauer and Rudebusch (2020) setup; the Bayesian estimation approach is explained in Section 3. Section 4 presents the empirical results; first for the United States in Section 4.2 and then for the euro area in Section 4.3. Finally, Section 5 concludes.

2 The Model

2.1 A semi-structural macro model with a term structure

Our semi-structural macro-finance model incorporates a variation and extension of the approach by Holston et al. (2017) (henceforth HLW), which is in turn based on Laubach and Williams (2003).[7] The model is in discrete time and in our econometric set-up one period corresponds to one quarter. The real rate gap, the output gap and inflation interact through backward-looking IS and Phillips curves. In the IS curve

$$\tilde{x}_t = a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \frac{a_3}{2}\left(\tilde{r}_{t-1} + \tilde{r}_{t-2}\right) + \varepsilon_t^{\tilde{x}}, \tag{1}$$

the output gap $\tilde{x}_t$ is defined as $\tilde{x}_t = x_t - x_t^*$, with x_t and x_t^* denoting log actual and log potential output, respectively, and the real interest rate gap $\tilde{r}_t = r_t - r_t^*$ is the difference between the actual real rate r_t and its natural counterpart r_t^*. Potential output x_t^* evolves according to

$$x_t^* = x_{t-1}^* + g_{t-1} + \varepsilon_t^{x^*}, \tag{2}$$

where g_t is the expected quarterly growth rate of potential output and $\varepsilon_t^{x^*}$ captures the unexpected part of potential growth. The real natural rate r_t^* is the sum of the annualized expected growth rate of potential output and a "catch-all", non-growth component, denoted z_t, i.e.

$$r_t^* = 4g_t + z_t. \tag{3}$$

Both g_t and z_t follow a random walk

$$g_t = g_{t-1} + \varepsilon_t^g, \qquad \text{and} \qquad z_t = z_{t-1} + \varepsilon_t^z. \tag{4}$$

[7] Their model extends the unobserved components model by Clark (1987), decomposing macroeconomic variables into random-walk trends and stationary cycles.

The z_t component captures effects such as saving-investment imbalances arising from longer retirement periods, as well as an increased demand for safe assets, (Del Negro et al., 2017, 2019), or other financial frictions.

For measuring actual real rates, the observed short-term nominal interest rate needs to be deflated by a measure of expected inflation. Laubach and Williams (2003) proxy inflation expectations by forecasts from an AR(3) estimated over a rolling window and HLW use a trailing four-quarter average of inflation to approximate inflation expectations and construct ex ante real rates. By contrast, we define the ex ante real rate in a model-consistent manner as

$$r_t = i_t - E_t \pi_{t+1}, \tag{5}$$

where i_t denotes the nominal short-term interest rate and $E_t \pi_{t+1}$ is the conditional expectation of next period's inflation based on model dynamics.[8]

Our second main equation, the Phillips curve, is given by

$$\tilde{\pi}_t = b_1 \tilde{\pi}_{t-1} + b_2 \tilde{x}_{t-1} + \varepsilon_t^{\pi}, \tag{6}$$

where $\tilde{\pi}_t = \pi_t - \pi_t^*$, represents the inflation gap, i.e. the difference of inflation π_t from its trend π_t^* that is also assumed to follow a random walk

$$\pi_t^* = \pi_{t-1}^* + \varepsilon_t^{\pi^*}. \tag{7}$$

As a result, the real rate gap $\tilde{r}_t$ affects – via the output gap – the cyclical component of inflation. This specification differs from Laubach and Williams (2003) and HLW who also impose a unit root on inflation, but eschew an explicit expression for its stochastic trend. Specifically, their Phillips curve is formulated for inflation in levels (rather than inflation gaps) and coefficients of lagged inflation terms are constrained to sum to unity.

We close the model by specifying the dynamics of the nominal risk-free yield curve. At each point in time, the cross section of yields of all maturities is assumed to be explained by three factors ('level', L_t, 'slope', S_t, and 'curvature', C_t) with factor loadings across maturities following the functional form of Nelson and Siegel (1987):

$$y_t(\tau) = \mathcal{A}(\tau) + L_t + \theta_s(\tau) S_t + \theta_c(\tau) C_t \tag{8}$$

where $y_t(\tau)$ denotes the τ-quarter bond yield, and factor loadings are given by $\theta_s(\tau) = \frac{1-\exp(-\lambda\tau)}{\lambda\tau}$

[8] For more details, see Annex A.

and $\theta_c(\tau) = \frac{1-\exp(-\lambda\tau)}{\lambda\tau} - \exp(-\lambda\tau)$.

An increase in the level factor induces a parallel upward shift of the whole yield curve, an increase in the slope factor increases the short end by more than the long end (hence, strictly speaking, 'negative slope factor') and an increase in the curvature factor accentuates the curvature at short- to medium-term maturities. The parameter λ governs how strongly a change in the slope factor S_t affects the slope of the yield curve and at which maturity the curvature factor has its maximum impact on the yield curve.

The intercept term $\mathcal{A}(\tau)$ does not appear in the original Nelson-Siegel specification. It is added to rule out arbitrage, as detailed further in Appendix B. Besides depending on maturity, $\mathcal{A}(\tau)$ is a function of the Nelson-Siegel factor loadings as well as of factor innovation variances.

If yield factor dynamics were constrained to be stationary, all yields would converge to a constant mean. In particular, this convergence would imply that the long-horizon expectation of the nominal one-period rate $i_t \equiv y_t(1)$ is constant, i.e. $i_t^* \equiv \lim_{h\to\infty} E_t i_{t+h} = i^*$. However, as our macro module specifies integrated processes for trend inflation and the natural real rate, the long-run Fisher equation, $i_t^* = \pi_t^* + r_t^*$, implies time-variation in the attractor for the nominal short-term rate. We incorporate this time-variation by allowing the level factor to be non-stationary, while imposing stationarity on the slope and curvature factors. Specifically, we decompose the level factor as

$$L_t = L_t^* + \tilde{L}_t \tag{9}$$

where L_t^* is a non-stationary trend such that $\lim_{h\to\infty} E_t L_{t+h}^* = L_t^*$ and $\tilde{L}_t$ is a zero-mean stationary (or "cyclical") component. From (8), for the one-quarter short-term interest rate we have

$$i_t = \mathcal{A}(1) + L_t + \theta_s(1)S_t + \theta_c(1)C_t \tag{10}$$

and hence for the limit

$$\lim_{h\to\infty} E_t i_{t+h} \equiv i_t^* = \mathcal{A}(1) + L_t^* + \theta_s(1)\bar{S} + \theta_c(1)\bar{C}, \tag{11}$$

where $\bar{S}$ and $\bar{C}$ denote the constant long-run means of the slope and curvature factor, respectively. In combination with equation (11), the long-run Fisher equation $i_t^* = \pi_t^* + r_t^*$ pins down the trend component of the level factor as $L_t^* = \pi_t^* + r_t^* - \theta_s(1)\bar{S} - \theta_c(1)\bar{C} - \mathcal{A}(1)$. As L_t^* is a latent process and $\mathcal{A}(1)$ is a free parameter (see Appendix B) we set $\mathcal{A}(1) = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ so that the long-run level factor is equal to the nominal short-term natural rate

$$L_t^* = i_t^* \equiv r_t^* + \pi_t^*. \tag{12}$$

For the stationary zero-mean component of the level factor we specify an AR(1) process

$$\tilde{L}_t = a_L \tilde{L}_{t-1} + \varepsilon_t^{\tilde{L}},$$

with $|a_L| < 1$. Finally, slope S_t and curvature C_t are assumed to follow a bivariate, stationary VAR that also includes the inflation and output gap as potential drivers:

$$\begin{aligned} S_t &= a_{10} + a_{11} S_{t-1} + a_{12} C_{t-1} + a_{13} \tilde{\pi}_{t-1} + a_{14} \tilde{x}_{t-1} + \varepsilon_t^S, \\ C_t &= a_{20} + a_{21} S_{t-1} + a_{22} C_{t-1} + a_{23} \tilde{\pi}_{t-1} + a_{24} \tilde{x}_{t-1} + \varepsilon_t^C. \end{aligned}$$

Our model implies a "natural yield curve" at each point in time, i.e. a set of attractors for all maturities. Taking limits on equation (8),

$$\lim_{h \to \infty} E_t y_{t+h}(\tau) \equiv y_t(\tau)^* = \mathcal{A}(\tau) + L_t^* + \theta_s(\tau)\bar{S} + \theta_c(\tau)\bar{C} \quad \text{for all } \tau. \tag{13}$$

The location of the natural yield curve varies over time with the stochastic drift in the level factor that is, according to equation (12), pinned down by the natural real short-term rate and trend inflation. At the same time, slope and curvature converge to constant means implying that the long-run *shape* of the natural yield curve is time-invariant, while the long-run *level* can change. In particular, the "natural yield spread" or slope

$$y_t^*(\tau) - y_t^*(1) = \mathcal{A}(\tau) + \theta_s(\tau)\bar{S} + \theta_c(\tau)\bar{C} \tag{14}$$

is time invariant. L_t^* cancels from the slope expression: the short-term natural real rate and trend inflation equally affect the short and the long end of the natural yield curve.

We compute the model-consistent term premium of maturity τ, $TP_t(\tau)$, as the difference between the model-implied τ-period bond yield and its expectations component, i.e. the expected average of future short rates over the respective maturity:

$$TP_t(\tau) = y_t(\tau) - \frac{1}{\tau} \sum_{h=0}^{\tau-1} E_t(i_{t+h}). \tag{15}$$

For computing the expectations component recall from (10) that the nominal short-term rate is a linear function of level, slope and curvature, where the level is in turn linked to the natural real rate and the inflation trend. Given the dynamics of the yield curve factors model-consistent expectations $E_t(i_{t+h})$ can be computed for all relevant horizons h.

Like the slope, also the term premium is stationary and converges to a constant mean, i.e.

the term structure of term premia has a time-invariant attractor. Expanding the expression for the term premium in (15), we have

$$
\begin{aligned}
TP_t(\tau) &= \mathcal{A}(\tau) + i_t^* + \tilde{L}_t + \theta_s(\tau) S_t + \theta_c(\tau) C_t && (16)\\
&- \frac{1}{\tau} \sum_{h=0}^{\tau-1} E_t \big[\mathcal{A}(1) + i_{t+h}^* + \tilde{L}_{t+h} + \theta_s(1) S_{t+h} + \theta_c(1) C_{t+h} \big]. && (17)
\end{aligned}
$$

Noting that since $E_t(i_{t+h}^*) = i_t^*$ for all h, the i^* terms cancel out from the above expression. Moreover, $E_t(\tilde{L}_{t+h})$, $E_t(S_{t+h})$ and $E_t(C_{t+h})$ are all independent of i_t^* or any trending variable. Hence, $\lim_{h\to\infty} E_t TP_{t+h}(\tau)$ is constant over time.

The stationarity of the slope of the yield curve and term premia differs from the setting of Bauer and Rudebusch (2020). In both models, the natural nominal short rate i_t^* serves as a stochastic trend for the level of the yield curve. However, in their set-up the natural nominal short rate affects also the slope and curvature, and term premia likewise incorporate a stochastic trend.

HLW treat the short-term real rate as an exogenous variable. Accordingly their model does not tie the evolution of the actual and natural real rate together, i.e. the real rate gap can arbitrarily widen.

By contrast, in our model the yield curve equations pin down the dynamics of the short-term nominal and real rate Our model, rendering the real rate gap $\tilde{r}_t = r_t - r_t^*$, stationary. Formally, we have

$$
\begin{aligned}
r_t - r_t^* &= i_t - E_t \pi_{t+1} - r_t^* \\
&= r_t^* + \pi_t^* + \tilde{L}_t - E_t(\tilde{\pi}_{t+1} + \pi_{t+1}^*) - r_t^* \\
&= \tilde{L}_t - E_t \tilde{\pi}_{t+1}
\end{aligned}
$$

i.e. the real rate gap is the difference between the cyclical components of the yield curve level factor and inflation. As both of them are stationary mean-zero processes, $\lim_{h\to\infty} E_t \tilde{r}_{t+h} = 0$ at any point in time. In other words, while the actual real rate and its natural counterpart are both integrated processes, they share the same stochastic trend, so they are cointegrated and their difference is stationary.

2.2 State-space representation

Writing all model equations in state-space representation, the state vector ξ_t comprises the term structure factors (cyclical level component, slope and curvature), trend inflation, potential output, expected potential output growth, the non-growth driver of the natural rate, the cyclical component of inflation, the output gap and some lagged variables (to cater for the dynamic structure of our model):

$$\xi_t = (\tilde{L}_t, S_t, C_t, \pi_t^*, x_t^*, g_t, z_t, \tilde{\pi}_t, \tilde{x}_t, \tilde{L}_{t-1}, S_{t-1}, C_{t-1}, \tilde{\pi}_{t-1}, \tilde{x}_{t-1})'.$$

Combining the IS curve (1), the Phillips (6) curve and the laws of motion for the latent variables and is of the following form:

$$\xi_t = \mu + \mathbf{F}\xi_{t-1} + \mathbf{G}e_t, \quad e_t \sim \mathcal{N}(0, \mathbf{I}), \tag{18}$$

For the measurement variables, we assume that (log) output and inflation are measured without error so that both are simply the sum of their respective trend and cyclical component:

$$x_t = x_t^* + \tilde{x}_t \tag{19}$$

$$\pi_t = \pi_t^* + \tilde{\pi}_t \tag{20}$$

We further include as measurement a set of zero-coupon bond yields of maturities ranging from τ_1=1 quarter to $\tau_K = 40$ quarters. Observed yields $y_t(\tau_i)$ equal their model-implied counterpart in (8) plus a measurement error

$$y_t(\tau_i) = \mathcal{A}(\tau_i) + \tilde{L}_t + L_t^* + \theta_s(\tau_i)S_t + \theta_c(\tau_i)C_t + u_t^{\tau_i}, \quad u_t^{\tau_i} \sim \mathcal{N}(0, \sigma_{\tau_i}^2), \quad i = 1, \ldots, K \tag{21}$$

Finally, we give the model a helping hand in identifying the latent variables by adding some survey information to our measurements. In this context we note that Kim and Wright (2005) and Geiger and Schupp (2018) have previously incorporated survey information into canonical term-structure models with the effect of informing the degree of mean reversion of model-based expected interest rates. Specifically, we include expectations of average inflation over long horizons $E_t^{surv}\pi_{t+\infty}$ from Consensus Economics. We match these expectations with the model's latent trend inflation plus a measurement error:

$$E_t^{surv}\pi_{t+\infty} = \pi_t^* + u_t^{s,\pi}, \quad u_t^{s,\pi} \sim \mathcal{N}(0, \sigma_{s,\pi}^2). \tag{22}$$

We also use survey information about near-term interest rate expectations. This approach helps us to get a better handle on model-implied short-rate expectations over short– to medium-term horizons and hence to obtain a more accurate grasp of corresponding term premia. Specifically, we match Consensus survey expectations of short-term rates four quarters ahead, $E_t^{surv} y_{t+4}(1)$, with the corresponding model-implied expectation plus a measurement error:

$$E_t^{surv} y_{t+4}(1) = \mathcal{A}(1) + E_t L_{t+4} + \theta_S(1) E_t S_{t+4} + \theta_C(1) E_t C_{t+4} + u_t^{s,sr}, \quad u_t^{s,sr} \sim \mathcal{N}(0, \sigma_{s,sr}^2). \tag{23}$$

Additionally, for the euro-area version of the model (not for the US), we include survey information about nominal interest rate expectations over longer horizons. This addition turned out to be necessary to better identify low-frequency movements in the natural rate of interest that, given the short sample available for the euro area with only few business cycles, are particularly challenging to filter out.

For the euro area we would ideally want to use long-horizon expectations of *short*-term interest rates constituting the direct survey counterpart to i^*, but such surveys are only available as of 2016. We therefore use long-horizon expectations of *long*-term interest rates that are available with a biannual frequency since at least 1995Q1, the start of our sample. These should also be informative about i^* because i^* constitutes the level of the complete far-ahead yield curve, but we need to take into account the relevant information about the slope of the yield curve to match the long-rate-long-horizon surveys with the model. Accordingly, we equate the survey expectation with the model-expectation of the ten-year rate plus a measurement error[9]:

$$\begin{aligned} E_t^{surv} y_{t+\infty}(40) &= \lim_{h\to\infty} E_t y_{t+h}(40) + u_t^{s,lr}, \\ &= \mathcal{A}(40) + L_t^* + \theta_S(40)\bar{S} + \theta_C(40)\bar{C} + u_t^{s,lr}, \end{aligned} \tag{24}$$

with $u_t^{s,lr} \sim \mathcal{N}(0, \sigma_{s,lr}^2)$. Finally, collecting the observed yields, output, inflation and surveys in the observation vector ζ_t,

$$\zeta_t = \left(y_t(\tau_1), \ldots y_t(\tau_K), x_t, \pi_t, E_t^{surv} y_{t+4}(1), E_t^{surv}\pi_t^*, E_t^{surv} y_{t+\infty}(40)\right)',$$

where the last element is absent for the US version. The measurement equation of the state space model can be represented as

$$\zeta_t = \gamma + \mathbf{C}\xi_t + \mathbf{D}u_t \qquad \text{with} \quad u_t \sim \mathcal{N}(0, \mathbf{I}). \tag{25}$$

Appendix A lists the structure of the system matrices of the state space model (25) and (28) in detail.

[9]The horizon asked in the Consensus Survey is 6 to 10 years ahead. We treat it as the horizon at which survey panelists assume that variables have essentially converged to their (possibly time-varying) long-run means, hence we equate the survey with the "infinite horizon" model counterpart.

3 Estimation

In our Bayesian approach to estimating the state-space model we rely, to a great extent, on largely uninformative priors. Our approach allows simultaneous estimation of all model parameters and thereby eschews the multi-step maximum-likelihood approach adopted by Holston et al. (2017). As common in Bayesian estimation of unobserved components models, we use conjugate priors, the Gibbs sampler and the Durbin and Koopman (2002) simulation smoother to jointly estimate potential output growth, output gaps, trend inflation and real equilibrium interest rates for the United States and the euro area. The simulation smoother is initialized using HP-filtered trends and OLS estimates for parameters. The exception is the Nelson-Siegel parameter λ that we calibrate outside the Bayesian framework by estimating a yields-only Dynamic Nelson-Siegel (DNS) model in the spirit of Diebold and Li (2006) using maximum likelihood and the Kalman filter. Including survey data creates missing observations in the measurement equation, because some of the surveys start only after the start of the sample and some surveys are initially only available biannually. We therefore adapt the Durbin and Koopman simulation smoother to allow for mixed frequencies and treat missing values as unobserved variables.[10]

The model is estimated using quarterly data. Appendix C describes the data in detail. We estimate the US version of the model over the sample period 1961Q2–2019Q4 and the euro area version over the period 1995Q1–2019Q4. As the euro was introduced in 1999, we use synthetic data, i.e. aggregates of individual country data for the four years prior to 1999. However, we decided to not go back as far as HLW who start in 1972: with separate monetary policies across countries our linking of (synthetic aggregated) macro data and (synthetic aggregated) yield data could lead to results that are difficult to interpret economically; and consistent euro-area yield curve data are not available back into the 1970s.

4 Results

4.1 Parameter estimates

Table 1 presents posterior means of parameter estimates from the macro block of the model and compares them with those published by HLW. Even though parameter estimates are broadly consistent, the two studies differ across several dimensions, including Bayesian vs. multi-step ML estimation, closing the model with nominal yield dynamics or not, specification of inflation dynamics, using survey information or not, different samples.

The loading coefficients of the real rate gap, a_3, in the IS equation and of the output gap,

[10] See Durbin and Koopman (2012), pp. 110-112, for details.

Table 1: Prior and posterior densities of parameter estimates

		Prior		US posterior					EA posterior				
	Distr.	P1	P2	Mean	Median	5%	95%	HLW	Mean	Median	5%	95%	HLW
a_1	$\mathcal{N}$	1.5	0.5	1.76	1.76	1.63	1.86	1.55	1.6	1.61	1.39	1.79	1.67
a_2	" $\mathcal{N}$"	-0.6	0.5	-0.8	-0.8	-0.9	-0.67	-0.61	-0.65	-0.66	-0.83	-0.45	-0.72
$\frac{a_3}{2}$	" $\mathcal{N}$"	-0.1	0.05	-0.01	-0.01	-0.03	0	-0.06	-0.01	-0.01	-0.03	0	-0.04
b_1	" $\mathcal{N}$"	0.6	1	0.8	0.83	0.54	0.92	0.67	0.81	0.82	0.65	0.92	0.69
b_2	" $\mathcal{N}$"	0.15	0.05	0.13	0.12	0.01	0.27	0.08	0.06	0.05	0	0.15	0.06
a_L	" $\mathcal{N}$"	0.5	0.1 (0.025)	0.67	0.75	0.21	0.96		0.94	0.94	0.88	0.99	
$\sigma^2_{L^c}$	Γ^{-1}	4	2	0.39	0.38	0.27	0.56		0.24	0.24	0.19	0.33	
σ^2_S	$\mathcal{W}^{-1}$	10	$10 \cdot \mathbf{I_2}$	1.84	1.7	1.04	3.56		0.36	0.34	0.25	0.61	
σ^2_C				5.73	5.66	4.69	7.1		1.81	1.65	0.93	3.7	
$\sigma^2_{\bar{\pi}^*}$	Γ^{-1}	4	2	0.04	0.04	0.03	0.05		0.04	0.04	0.03	0.06	
$\sigma^2_{x^*}$	Γ^{-1}	4	2	0.43	0.4	0.32	0.65	0.34	0.14	0.13	0.09	0.24	0.20
σ^2_g	Γ^{-1}	14	0.02	0.001	0.001	0.0007	0.002	0.01	0.001	0.001	0.0006	0.0016	0.00
σ^2_z	Γ^{-1}	4	2	0.11	0.1	0.07	0.21	0.04	0.07	0.07	0.05	0.09	0.05
$\sigma^2_{\tilde{\pi}}$	Γ^{-1}	4	2	0.44	0.42	0.32	0.65	0.64	0.14	0.14	0.1	0.18	0.94
$\sigma^2_{\tilde{x}}$	Γ^{-1}	4	2	0.13	0.11	0.07	0.25	0.11	0.13	0.13	0.08	0.19	0.08
$\sigma^2_{s,\pi}$	Γ^{-1}	4	2	0.02	0.02	0.02	0.03		0.03	0.03	0.02	0.05	
$\sigma^2_{s,shsr}$	Γ^{-1}	4	2	0.02	0.02	0.01	0.04		0.29	0.27	0.1	0.72	
$\sigma^2_{s,lhlr}$	Γ^{-1}	4	2	0	0	0	0		0.04	0.04	0.03	0.06	

Note: The table shows prior and posterior moments of the structural model parameters, based on 100,000 iterations of the Gibbs sampler of which we discarded the first 90,000 draws and subsequently kept each tenth draw. Convergence is checked on the basis of recursive means as proposed by Geweke (1991). To ensure that the loadings of the real rate gap and the output gap in the IS curve and the Phillips curve, respectively, have the economically 'correct' sign, we discard those parameter draws in sampling from the posterior that violate those sign conditions. The first and second prior parameters, P1 and P2, equal the mean and variance of the distribution in case of the Normal distribution, and shape and scale in case of either inverse gamma or inverse Wishart distribution. HLW refers to the published estimates from Holston et al. (2017) from the New York Fed. Most inverse gamma priors for the variances are based on the Γ^{-1}(4,2) parameterisation, which implies a mean of 0.66 and a standard deviation of 0.47. We use conjugate priors for all model parameters and variances, i.e. prior distributions are either normal inverse gamma or normal inverse Wishart. All priors are uninformative with the exception of the variance of shocks to expected potential output growth σ^2_g. Here, we choose shape and scale parameters of the inverse gamma distribution such that the mean equals 0.0015 implying that, a priori, the variance of the change in (quarterly) potential output growth over one century equals 0.6%. Table 1 summarizes the priors of the main structural parameters.

b_2, in the Phillips curve are small. In particular, the estimated slope of the IS curve (for both the United States and the euro area) is below 0.1 (in absolute terms) – the critical threshold beneath which filtering uncertainty rises dramatically, as reported in Fiorentini et al. (2018). The corresponding IS curve estimates by HLW are a bit higher, yet not exceeding that threshold either. By contrast, our estimates of the slope of the Phillips curve b_2 are higher than in HLW and above 0.1, probably owing to differences in the Phillips-curve specifications (as discussed in Section 2.1). The variance of innovations to the Phillips curve, $\sigma^2_{\tilde{\pi}}$, is estimated to be much smaller, which is likely to reflect our explicit decomposition of inflation dynamics into a low-frequency stochastic trend and a stationary component. We also estimate the variance of shocks to the non-growth component, σ^2_z to be higher, especially for the US. Closing the model with nominal yield curve dynamics and rendering the real-rate gap stationary makes r^* track the real rate of interest more closely than in HLW. Accordingly the non-growth component needs to be able to capture a larger wedge between expected potential output growth and the trend in the natural real rate.

Finally, the estimated variance of measurement errors corresponding to the inflation surveys,

$\sigma^2_{s,\pi}$, is fairly tight (both for the United States and the euro area), hence keeping model-based long-run inflation expectations relatively close to their survey-based counterparts. A similar order of magnitude prevails for the measurement errors of long-rate-long-horizon interest rate surveys deployed for the euro area, while the short-rate-short-horizon interest rate surveys are matched with a larger measurement error on average – especially for the euro area. Nevertheless, surveys seem to be helpful for informing parameter estimates: estimating a euro-area specification for which the short-rate short-horizon surveys are dropped (but the rest of the specification remains the same) leads to a distinctly lower estimated persistence of short-rate dynamics, in turn implying excessively low and negative term premia.

4.2 United States

For the United States, Figure 2 displays inflation, the nominal short rate, and the real short rate together with their estimated trends. For all three variables, the model-implied trend is visibly in line with the low-frequency component of the corresponding observed variable. In particular, the natural real rate tracks the trend of the actual real rate, a feature consistent with the model-implied stationarity of the real rate gap.

Co-plotting our natural real rate trajectory with that by HLW in Figure 3 validates their estimate of a fall in r^* since the 1980s. Yet there are important differences. Until the 1980s, our natural rate estimates are by up to two percentage points lower than the ones reported by HLW and closer to the observed real interest rate, in their case implying a wider and more persistent real interest rate gap. Likewise, at the end of the sample, our natural rate estimates are by more than one percentage point lower than reported by HLW, with their gauge of r^* being a touch above zero while our estimate has fallen as low as up to minus one percent.

These discrepancies likely reflect that the specification by HLW does not impose the gap between the natural and actual real rate to be stationary. At the same time our r^* estimates during the 1970s are close to the average of real-time model estimates reported in Bauer and Rudebusch (2020).[11] In any event, estimates in the literature are diverging widely from each other, and there is a sizeable margin of model and specification uncertainty (besides estimation uncertainty for a given model).[12]

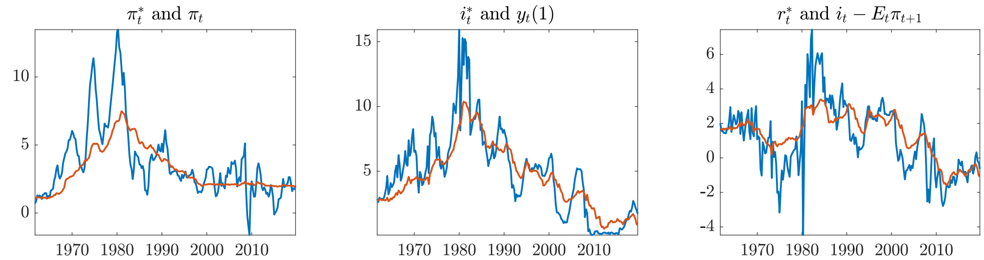
Finally, our posterior-based uncertainty bands (5% to 95% range) span around two percent-

[11] See their Figure 2, Panel B.

[12] See, e.g., the cross-model ranges provided in Bauer and Rudebusch (2020) or in Williams et al. (2017). The synopsis by Neri and Gerali (2019) focuses on natural-rate estimates obtained from structural (DSGE) models: for the time from 2010 to 2016 (when their sample ends) most of the reported US results are ranging distinctly sub-zero with some even well below minus two percent. Lopez-Salido et al. (2020) illustrate the sensitivity of results to model specification within the Holston et al. (2017) approach: depending on the way that short- and long-run inflation expectations enter the model, natural rate results can range between the reported positive levels and around minus one percent.

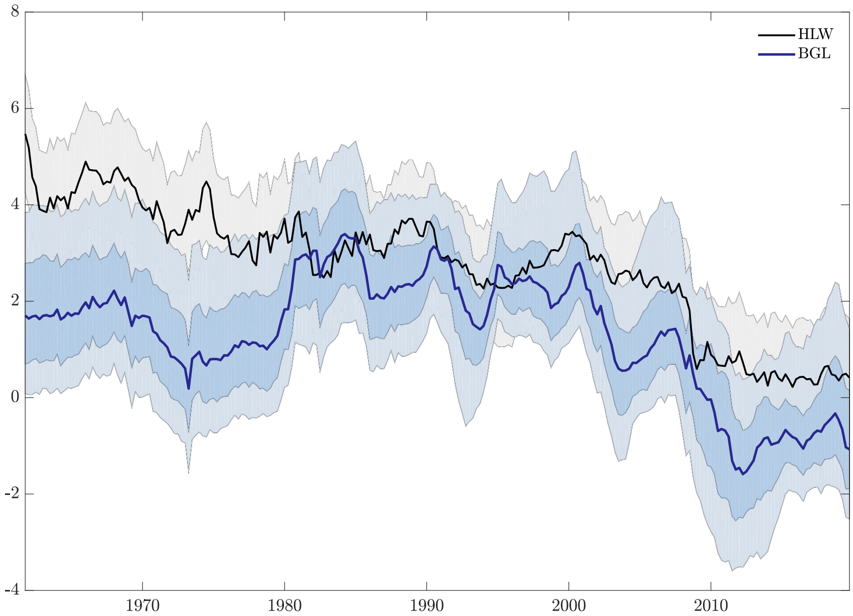
age points which is less than the c. 2.5 percentage points uncertainty band surrounding the HLW estimates based on a one standard error deviation band (corresponding to a 68 percentage point range for a normal distribution). However, ranges are not directly comparable as HLW only report one-sided filtering uncertainty, while our ranges reflect parameter and filtering uncertainty, throughout based on the full sample.

Figure 2: US macroeconomic variables and model-implied trends



Note: The figure shows the estimated trends (in red) and the observed macro-variables (in blue).

Figure 3: US natural rate estimates

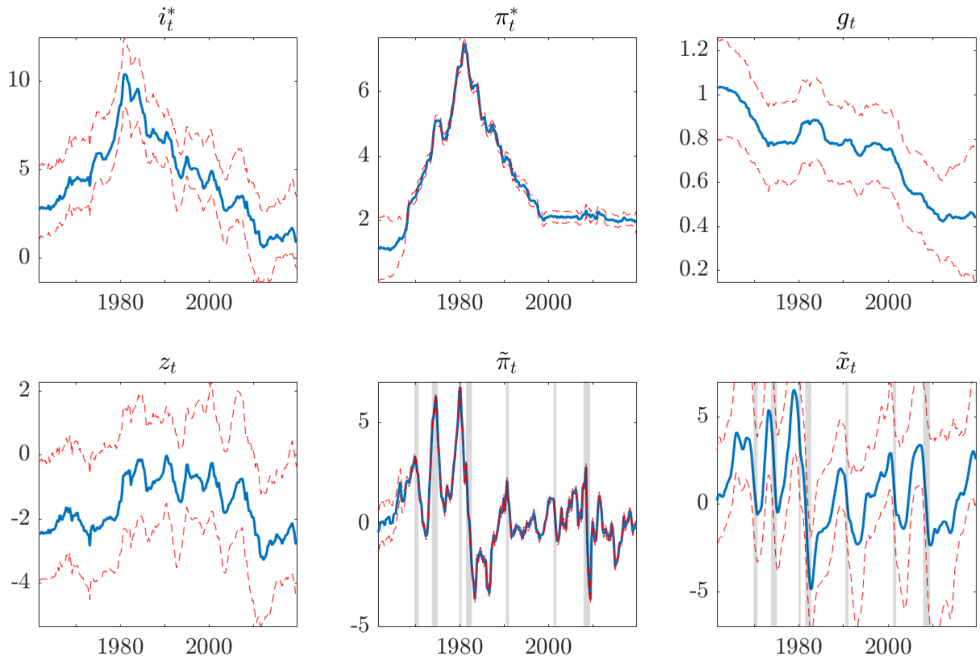


Note: The figure shows our natural rate estimate in blue with the and 5%, 16%, 84% and 95% percentiles depicted by the blue-shaded areas, respectively. For comparison, the black line depicts the official (one-sided) r_t^* estimate from Holston et al. (2017), which is updated by the New York Fed. The 68% confidence band (i.e. plus/minus one standard deviation) is based on the average standard error and depicted by the gray-shaded area.

As regards the other latent factors, our results suggest that (quarterly) potential real output growth fell over the sample period from 1.1% to around 0.4% in 2010 and has stayed low ever since, see Figure 4. Inflation and output gap estimates are reported together with NBER recessions. The cycles in these estimates match official recession dates rather well. Appendix D also shows broad consistency between our model-based output gap estimates with estimates by public policy agencies. Among the two components constituting the natural real rate, z_t and

g_t, the "catch-all" contribution z_t is less precisely estimated compared to potential growth (even when annualised). As shown in Fiorentini et al. (2018), the relatively high level of statistical uncertainty around the non-growth component of r^* can be traced back to weak loading coefficients of gap measures in the IS and the Phillips curves.

Figure 4: Further US latent macro variables

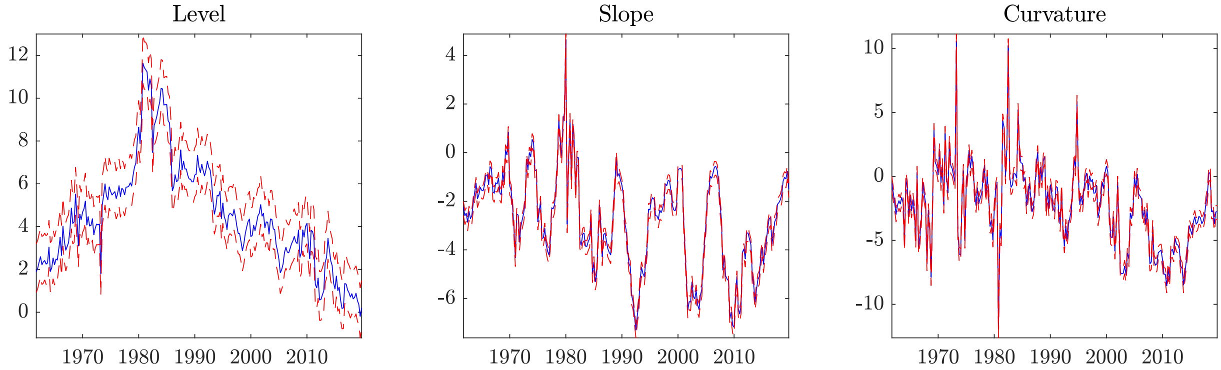


Note: The figure shows the estimated latent states of the model in blue together with their 5% and 95% percentiles in red-dashed. Shaded areas represent NBER recessions.

Figure 5 plots the AFNS yield curve factors together with their 5% and 95% percentiles. Three observations are in order. First, the path of these yield curve factors is close to what would be obtained from a yields-only dynamic Nelson-Siegel specification (not shown), which suggests a strong role for the cross-sectional yield curve information in pinning down these factors. Second, the statistical uncertainty in the estimation of the level factor L_t is sizeably higher than that of slope and curvature, possibly reflecting that the level factor plays a dual role: it is the time-varying anchor point for the whole term structure; and its trend component $L_t^* = i_t^* = r_t^* + \pi_t^*$ plays a key role in the determination of macroeconomic dynamics. Third, while the level factor exhibits a clear trend (sample autocorrelation of the estimated series equal to 0.956), the estimated slope and curvature appear rather mean reverting (autocorrelations of 0.877 and 0.617, respectively). This result supports – at least heuristically – our modeling choice of having only the level factor being driven by the stochastic trend but modeling slope and curvature as stationary.

Figure 6 displays decompositions of bond yields into the expectations component and the

Figure 5: US yield curve factors



Note: The figure shows the yield curve factors in blue with respective 5% and 95% percentiles in red-dashed.

term premium (see (15)). On the left Figure 6 shows our decomposition of the 5-year forward rate 5-years ahead into expectations (of the average short-term interest rate over that 5-year maturity horizon) and the term premium. While the expectations component exhibits a distinct rise and fall, the term premium estimates displays cyclical behaviour.

When plotting our term premia estimates against those commonly reported in the literature (yellow line versus grey range in Figure 6), the timing of troughs and peaks largely coincides. In particular, term premia have risen at the onset of the Global Financial Crisis and slumped with the start of the Federal Reserve's large-scale asset purchases at the end of 2008. They also display a sharp rise following the 'taper-tantrum' in 2013.[13]

Yet, while term-premia estimates from the literature display a distinct trend, especially for the long forward horizon, our term premia rather show cyclical dynamics, in line with economic theory, rising with the onset of economic downturns, and subsequently falling. This pattern reflects the underlying model mechanics: while the standard modeling approach is based on stationary factor dynamics and a time-invariant long-run mean for the short rate, our model features a time-varying attractor for the short-term interest rate. Accordingly, the expectations component is able to soak up a relatively larger part of the trend in long-term bond yields in our model, and the term premium does not have to incorporate a trend.

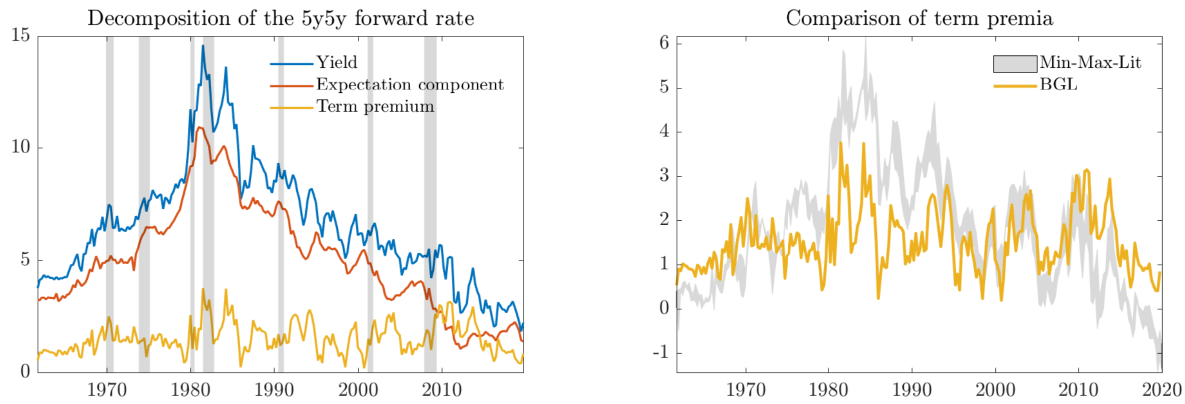
An important exception to findings of trending term premia (and a motivation for our work) is Bauer and Rudebusch (2020) who also incorporate a trend for long-horizon short-rate expectations. Their term premia estimates (blue and orange lines in Figure 7) for long-horizon forwards[14] do not show such a strong trend as the constant-mean models, but they are somewhat less cyclical than our term premia. This "inbetween" pattern of their estimates could arise from

[13]The 5-year, 5-year term premium increased markedly from 164bps to 204bps in 2013 Q2 (Fed Chair's Bernanke's speech was in May 2013).

[14]They report two sets of term premia, one based on a model where the shifting endpoint is taken as an observed (off-model) proxy, and one based on another model where the shifting endpoint is estimated within the model.

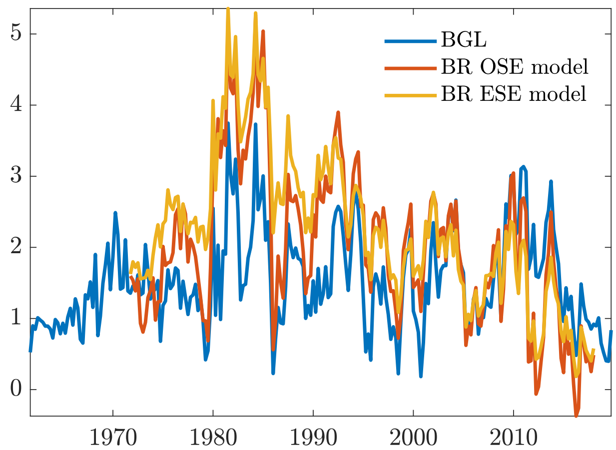
the fact that our model specification enforces stationary term premia by construction, while in their model the stochastic trend is allowed to also affect term premia.

Figure 6: Decomposition of US 5y5y rates



Note: The left figure shows the decomposition of the 5-year, 5-year forward yield in blue into the model-implied expectation component (red) and the term premium (yellow). Authors' calculations. NBER recessions in gray. The right figure compares our term premium estimate for the 5-year, 5-year forward bond yield with a min-max-range (grey area) of several estimates in the literature: Kim and Wright (2005) (taken from FRED), Adrian et al. (2013) and a DNS model following Diebold and Li (2006) (all authors' calculations).

Figure 7: Comparison to Bauer and Rudebusch (2020)



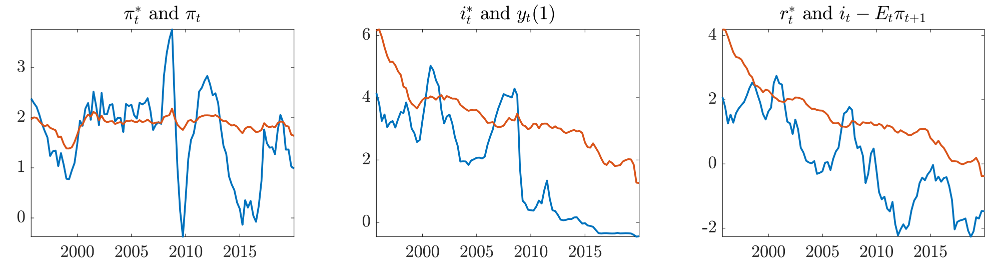
Note: The figure compares our term premium estimate of the 5-year, 5-year forward rate (in blue) with those presented in Bauer and Rudebusch (2020). OSE (in red) denotes the model with observed shifting endpoint, while ESE (in yellow) denotes the model with estimated shifting endpoint.

4.3 Euro area

For the euro area, Figure 8 shows inflation, the nominal short-term rate and the real rate together with their estimated trends. Trend inflation is relatively stable around 2%, in line with the evolution of the respective survey expectations and the fact that the measurement error for the survey is estimated to have a relatively small standard deviation (see Table 1). The natural nominal and real rate display a downward trend over the sample with the natural real rate having fallen to around zero percent and eventually into negative territory over the last few years. The corresponding HLW estimate of r^* follows a similar downward trend, but remains above zero at sample end, see Figure 9.

While the model specifies the real rate gap $(r - r^*)$ to be stationary, the realisation of the model-implied r and smoothed r^* in the right-panel of Figure 8 display a fairly protracted distance of both measures from each other for the euro area. For interpreting this outcome it is important to recall that the estimated r^* path is a result of both the stipulated model dynamics and the measurement variables. For the case at hand, the long-horizon survey expectations of long-term yields turn out to be particularly influential,[15] steering the inference about i^* and in turn (given the survey-aided π^* estimates) the natural rate estimate r^*. Dropping the long-horizon survey on long-term yields from the measurement variables would lower the corresponding i^* and r^* estimates and make the gap between r and r^* look more stationary. However, without the survey information, the (even) lower level of r^* and i^* at the sample end would imply extremely low long-horizon expectations of future nominal short-term rates, pushing the expectations component of long-term yields down to implausible levels: the expectations component of the 10-year interest rate at sample end would otherwise have amounted to -1%, below the lowest level of the short-term rate of interest observed so far in the euro area.[16] For these reasons we favour a specification with long-term interest rate survey expectations for the euro area, even if this choice comes at the cost of rendering the real rate gap less stationary – as reflected in our smoothed (small-sample) estimates.

Figure 8: Euro area macroeconomic observables and trends



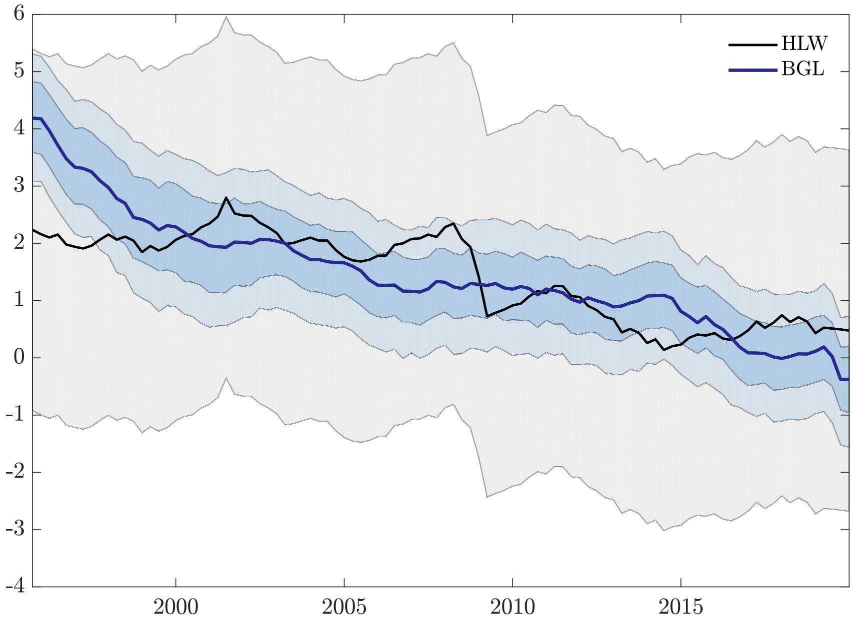
Note: The figure shows the estimated trends (in red) and the observed macro-variables (in blue). The black solid line reports our own Bayesian estimates of the specification in Holston et al. (2017).

Our uncertainty band around the estimated natural rate path in Figure 9 is considerably

[15] See especially the small measurement error variance in the last row of Table 1.

[16] By contrast, our current specification implies relatively plausible long-horizon expectations of short-term rates as is evident from comparing our estimates of i^* with long-horizon (six to ten years ahead) surveys of the *short-term* (three-month) interest rate. These survey data have not been used for the estimation as they are available only as of late 2016, so they can only serve as a yardstick for the end of the sample. These survey-implied rate expectations amount to around 2% until 2019, before declining by around half a percentage point, thus being close to our model estimates, compare our i^* estimates in the middle panel of Figure 8. If the model-implied inflation trend (a bit below 2% and in turn close to our survey variables) is considered to be reasonable as well, the proximity of i^* and π^* to their survey proxies suggests that our r^* estimates for the euro area over the last few years of the sample are also relatively reasonable.

Figure 9: Euro area natural rate estimates



Note: The figure shows our natural rate estimate in blue with 5%, 16%, 84% and 95% percentiles depicted by the blue-shaded areas. For comparison, the black line shows the (one-sided) estimate of Holston et al. (2017), which is updated by the New York Fed. The 68% confidence band (i.e. plus/minus one standard deviation) is based on the average standard error and depicted by the gray-shaded area.

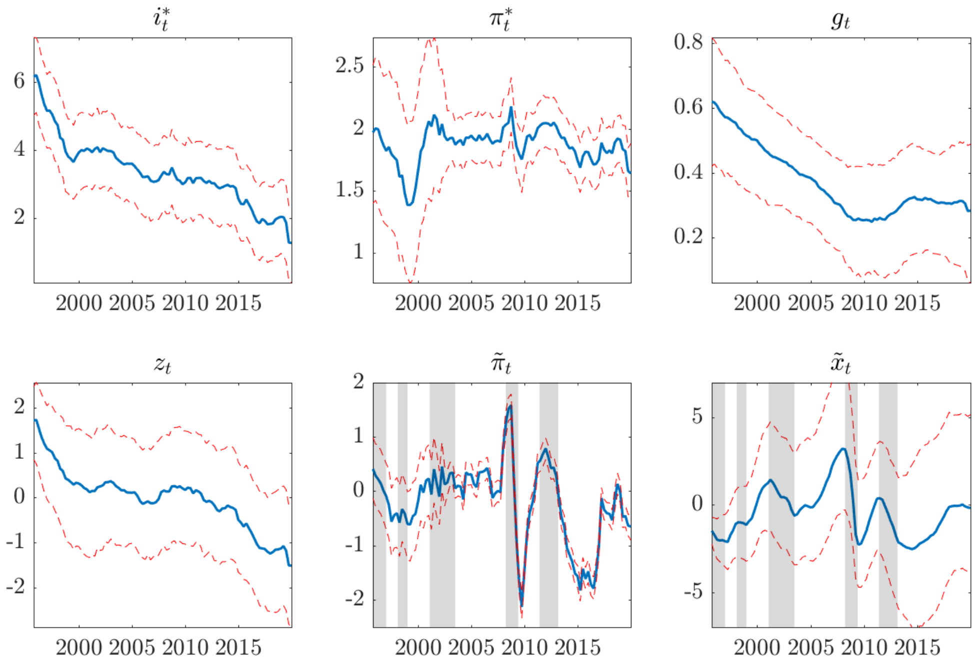
narrower than that around the HLW estimates. However, as discussed for the US case, the differences in modeling and estimation approaches imply a limited comparability of these bands.

As regards the other latent macroeconomic variables, Figure 10 shows that quarterly potential growth is estimated to have fallen from around 0.6% in the mid 1990s to about 0.2% in 2010, and recovering only marginally to 0.3% since then. Both the inflation and output gap show a consistent cyclical pattern, and the output gap estimate aligns relatively well with published estimates from the IMF and the European Commission (see Annex D).

Similar to the United States, the yield curve factors in the euro area line up tightly with their counterparts from a Nelson-Siegel model that is estimated solely with yield curve data, with the estimation uncertainty for the level factor being larger than for either the slope or curvature factor (see Figure 11). Again, the level factor displays a clear downward trend, in contrast to slope and curvature.

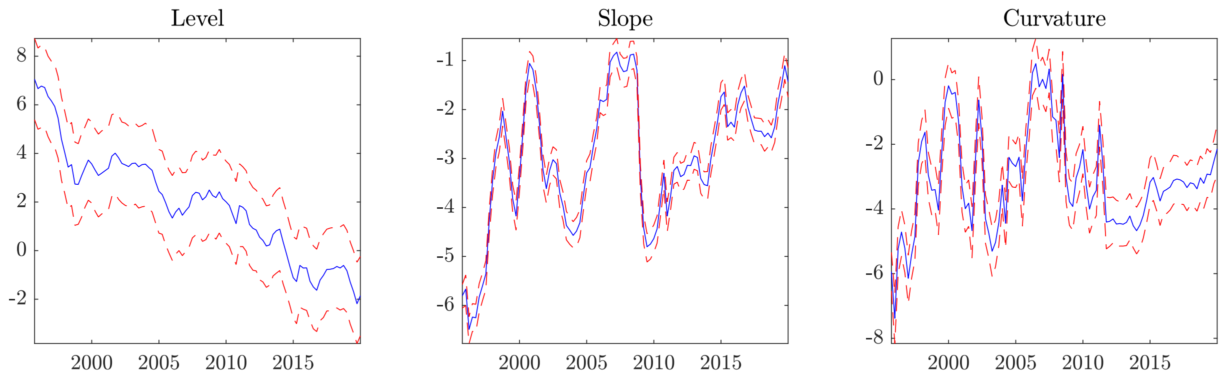
The left-hand side of Figure 12 shows the decomposition of 5-year-5-year forward rates into their expectations and term-premium components. It is primarily the expectations component that picks up the trend decline in interest rates, while the term premium exhibits a much less pronounced fall. The fall in the expectations component explains a large part of the fall in yields prior to the introduction of the euro in 1999 and after global financial crisis of 2008. During the intervening years our model attributes most of the falling trend in yield to the term premium, in line with commonly available estimates from the literature and as shown on the right-hand side

Figure 10: Euro area latent macro variables



Note: The figure shows the estimated latent states of the model in blue together with their 5% and 95% percentiles in red. Shaded areas represent CEPR recessions.

Figure 11: Euro area yield curve factors



Note: The figure shows the yield curve factors in blue with respective 5% and 95% percentiles in red. For comparison, the black-dashed lines represent yield curve factors obtained from a yields-only AFNS model.

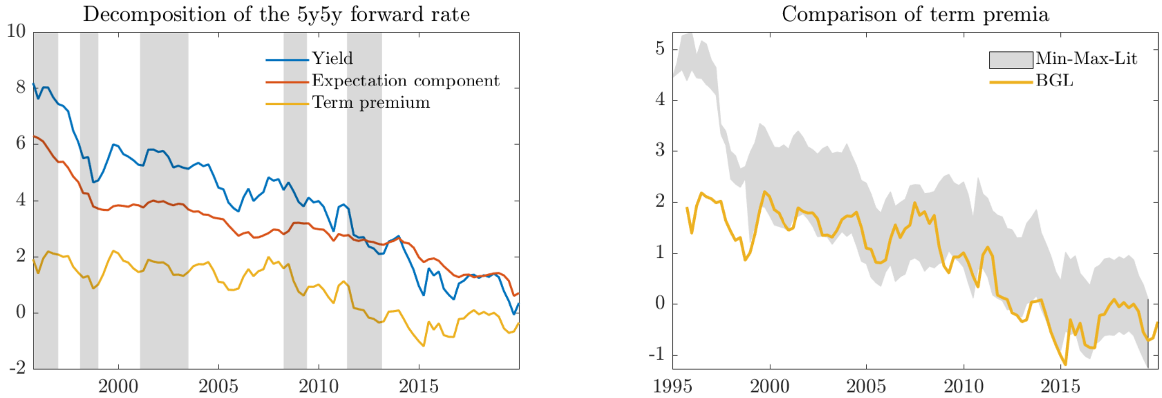
of Figure 12. Specifically, our estimates also posit a clear decline in term premia during summer 2014, when market expectations were intensifying that the ECB will embark on a major asset purchase program.[17]

5 Conclusion

In this paper, we join two strands of the literature: arbitrage-free models of bond yield dynamics incorporating a time-varying attractor ("shifting end point") for short rate expecta-

[17] See also Lemke and Werner (2020).

Figure 12: Decomposition of euro area 5y5y rates



Note: The left figure shows the decomposition of the 5-year, 5-year forward bond yield (blue) into the model-implied expectation component (red) and the term premium (yellow). Authors' calculations. Shaded areas represent CEPR recessions. The right figure compares our term premium estimate for the 5-year, 5-year forward bond yield with a min-max-range of several estimates in the literature, including estimates from Geiger and Schupp (2018), and estimates from Adrian et al. (2013) and Diebold and Li (2006) (both own estimates).

tions over long horizons – most recently exemplified by the frontier contribution of Bauer and Rudebusch (2020); and semi-structural macro models inferring the location and dynamics of the natural real rate of interest – the most prominent example being Holston et al. (2017). Our proposed model captures the joint dynamics of key macroeconomic variables following Holston et al. (2017). Different from Bauer and Rudebusch (2020), we do not treat the short-term nominal interest rate as exogenous, but rather endogenise it by modeling its dynamics as part of a complete arbitrage-free specification of the term structure. The nexus between the macro and the term structure building blocks of our model is the natural real rate. Relative to its position the actual real rate drives the business cycle; at the same time – together with trend inflation – it constitutes the underlying trend of the level of the yield curve.

Paired with a Bayesian estimation approach, our framework allows for simultaneous estimation of key unobservable macro objects like the natural real rate of interest, trend inflation and the output gap, as well as unobservable term premia incorporated in long-term bonds. The joint estimation and quantification of uncertainty distinguishes our method from most other studies in the aforementioned literature that tend to rather rely on multi-step approaches or treating estimates of latent factors as observables.

Consistent with Bauer and Rudebusch (2020), we find that taking into account the secular fall in equilibrium rates, term premia exhibit cyclical behavior over the business cycle, rather than the trend decline reported when using term structure models with a constant steady state.

We validate evidence of a recent decline in the natural rate of interest in advanced economies to levels around zero or into negative territory as reported, e.g. in Fiorentini et al. (2018); Gourinchas and Rey (2019); Jorda and Taylor (2019); Kiley (2020). But our estimates of the natural real rate deviate at times from those reported in Holston et al. (2017), inter alia due to our closing of the model via the yield curve dynamics and due to our inclusion of interest rate surveys.

Our model makes strides towards a better integration of macro and yield curve dynamics, here with a focus on the natural real rate of interest. Yet, two important further challenges require further research. First, incorporating the effective-lower bound constraint on interest rates and easing effects of central-bank asset purchase programs into our new macro-finance framework: Within the commonly used semi-structural approach (without yield curve) González-Astudillo and Laforte (2020) use information in long-term yields in estimating r^* to deal with the lower-bound constraint – but incorporating a suitable non-linear approach in a framework seeking to decompose yields into expectations and term premia remains an additional challenge. Second, updating our proposed modelling framework with data covering the pandemic crisis: Lenza and Primiceri (2020) argue that, in a VAR context, such data are better discounted for estimation, but not to be ignored for forecasting. In our context, we would observe that an update would not be necessary for model estimation, but extremely volatile data during the pandemic are bound to translate into large and equally volatile gyrations in filtered r^* estimates, making the interpretation of such estimates extremely challenging. A robust approach to discounting the impact of extreme data volatility on filtered r^* estimates is therefore still to be developed.

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Annex

A The state space model

The observation equations are given by:

$$
\begin{aligned}
y_t(\tau_i) =& \mathcal{A}(\tau_i) + \tilde{L}_t + L_t^* + \theta_s(\tau_i)S_t + \theta_c(\tau_i)C_t + u_t^{\tau_i}, \quad u_t^{\tau_i} \sim \mathcal{N}(0, \sigma_{\tau_i}^2), \quad i = 1, \ldots, K \\
x_t =& x_t^* + \tilde{x}_t, \\
\pi_t =& \pi_t^* + \tilde{\pi}_t, \\
E_t^{surv} y_{t+4}(1) =& \mathcal{A}(1) + E_t L_{t+4} + \theta_s(1) E_t S_{t+4} + \theta_c(1) E_t C_{t+4} + u_t^{s,sr}, \quad u_t^{s,sr} \sim \mathcal{N}(0, \sigma_{s,sr}^2) \\
E_t^{surv} \pi_{t+\infty} =& \pi_t^* + u_t^{s,\pi}, \quad u_t^{s,\pi} \sim \mathcal{N}(0, \sigma_{s,\pi}^2), \\
E_t^{surv} y_{t+\infty}(40) =& \mathcal{A}(40) + \theta_s(40)\bar{S} + \theta_c(40)\bar{C} + L_t^* + u_t^{s,lr}, \quad u_t^{s,lr} \sim \mathcal{N}(0, \sigma_{s,lr}^2)
\end{aligned}
$$

where $\theta_s(\tau)$ and $\theta_c(\tau)$ are the Nelson-Siegel loadings defined in the main text, $\mathcal{A}(\tau)$ is defined in the next section, and $L_t^* = r_t^* + \pi_t^*$. We allow for different measurement error variances across observed yields, but assume the one-quarter short-term rate $i_t \equiv y_t(1)$ to be matched without error, i.e. $\sigma_{\tau_1}^2 = 0$. The last three equations describe how surveys are mapped into unobserved trends subject to a measurement error. First, we use Consensus expectations of the short rate in one year time to inform our econometric model about the speed at which the short rate convergences to its time-varying attractor i_t^*. Second, we use Consensus expectations of long-term inflation as a noisy measure of trend inflation. Lastly, we incorporate long-term Consensus expectations of the 10-year rate 6-10 years in the future, denoted $\hat{E}_t^{surv} y_{t+\infty}(40)$, to inform estimation of L_t^*. As explained in the main text, treating the 6-10 year horizon as 'very long' the model-implied counterpart to the survey data is $\mathcal{A}(40) + \theta_s(40)\bar{S} + \theta_c(40)\bar{C} + L_t^*$.

The state equations are given by:

$$
\begin{aligned}
\tilde{L}_t &= a_L \tilde{L}_{t-1} + \varepsilon_t^{\tilde{L}}, \\
S_t &= a_{10} + a_{11} S_{t-1} + a_{12} C_{t-1} + a_{13} \tilde{\pi}_{t-1} + a_{14} \tilde{x}_{t-1} + \varepsilon_t^S, \\
C_t &= a_{20} + a_{21} S_{t-1} + a_{22} C_{t-1} + a_{23} \tilde{\pi}_{t-1} + a_{24} \tilde{x}_{t-1} + \varepsilon_t^C, \\
\pi_t^* &= \pi_{t-1}^* + \varepsilon_t^{\pi^*}, \\
x_t^* &= x_{t-1}^* + g_{t-1} + \varepsilon_t^{x^*} \\
g_t &= g_{t-1} + \varepsilon_t^g, \\
z_t &= z_{t-1} + \varepsilon_t^z, \\
\tilde{\pi}_t &= b_1 \tilde{\pi}_{t-1} + b_2 \tilde{x}_{t-1} + \varepsilon_t^{\pi},
\end{aligned}
$$

$$\tilde{x}_t = a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \frac{a_3}{2}\big(\tilde{r}_{t-1} + \tilde{r}_{t-2}\big) + \varepsilon_t^{\tilde{x}}.$$

Given that both the inflation $\tilde{\pi}_t$ and output gap $\tilde{y}_t$ are mean-zero by construction, we have

$$\begin{pmatrix} \bar{S} \\ \bar{C} \end{pmatrix} = \left(\mathbf{I_2} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)^{-1} \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}.$$

To calculate the real rate, $r_t = i_t - E_t\pi_{t+1}$, we assume expectations to be model-consistent. Taking conditional expectations of (6) gives

$$E_t\pi_{t+1} = E_t[\pi^*_{t+1} + \tilde{\pi}_{t+1}] = \pi^*_t + b_1\tilde{\pi}_t + b_2\tilde{x}_t.$$

Substitution yields

$$r_t \equiv y_t(1) - E_t\pi_{t+1} = y_t(1) - \pi^*_t - b_1\tilde{\pi}_t - b_2\tilde{x}_t.$$

Using equation (8) and $\mathcal{A}(1) = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ the real rate gap is given by

$$\begin{aligned} \tilde{r}_t =& r_t - r^*_t \\ =& y_t(1) - \pi^*_t - b_1\tilde{\pi}_t - b_2\tilde{x}_t - r^*_t \\ =& r^*_t + \pi^*_t + \tilde{L}_t + \theta_s(1)[S_t - \bar{S}] + \theta_c(1)[C_t - \bar{C}] - \pi^*_t - b_1\tilde{\pi}_t - b_2\tilde{x}_t - r^*_t \\ =& \tilde{L}_t + \theta_s(1)[S_t - \bar{S}] + \theta_c(1)[C_t - \bar{C}] - b_1\tilde{\pi}_t - b_2\tilde{x}_t. \end{aligned} \tag{26}$$

Finally, substituting the latter equation into the IS curve, we have

$$\begin{aligned} \tilde{x}_t =& a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \frac{a_3}{2}\big(\tilde{r}_{t-1} + \tilde{r}_{t-2}\big) + \varepsilon_t^{\tilde{x}} \\ =& a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \frac{a_3}{2}\big(\tilde{L}_{t-1} + \theta_s(1)[S_{t-1} - \bar{S}] + \theta_c(1)[C_{t-1} - \bar{C}] - b_1\tilde{\pi}_{t-1} - b_2\tilde{x}_{t-1}\big) \\ &+ \frac{a_3}{2}\big(\tilde{L}_{t-2} + \theta_s(1)[S_{t-2} - \bar{S}] + \theta_c(1)[C_{t-2} - \bar{C}] - b_1\tilde{\pi}_{t-2} - b_2\tilde{x}_{t-2}\big) + \varepsilon_t^{\tilde{x}} \\ =& \big(a_1 - \frac{a_3 b_2}{2}\big)\tilde{x}_{t-1} + \big(a_2 - \frac{a_3 b_2}{2}\big)\tilde{x}_{t-2} \\ &+ \frac{a_3}{2}\big(\tilde{L}_{t-1} + \theta_s(1)[S_{t-1} - \bar{S}] + \theta_c(1)[C_{t-1} - \bar{C}] - b_1\tilde{\pi}_{t-1}\big) \\ &+ \frac{a_3}{2}\big(\tilde{L}_{t-2} + \theta_s(1)[S_{t-2} - \bar{S}] + \theta_c(1)[C_{t-2} - \bar{C}] - b_1\tilde{\pi}_{t-2}\big) + \varepsilon_t^{\tilde{x}}. \end{aligned}$$

In compact state-space representation, the model can be written as[18]

$$\zeta_t = \gamma + \mathbf{C}\xi_t + \mathbf{D}u_t \qquad \text{with} \quad u_t \sim \mathcal{N}(0, \mathbf{I}) \tag{27}$$

[18]For the US version of the model, for which we do not use long-horizon/long-rate surveys, the last measurement equation is absent and dimensions adjust accordingly.

$$\xi_t = \mu + \mathbf{F}\xi_{t-1} + \mathbf{G}e_t \qquad \text{with} \quad e_t \sim \mathcal{N}(0, \mathbf{I}), \tag{28}$$

where

$$\zeta_t = \Big(y_t(\tau_1) \quad \dots \quad y_t(\tau_K) \quad x_t \quad \pi_t \quad E_t^{surv} y_{t+4}(1) \quad E_t^{surv}\pi_{t+\infty} \quad E_t^{surv} y_{t+\infty}(40) \Big)',$$

and

$$\xi_t = \Big(\tilde{L}_t \quad S_t \quad C_t \quad \pi_t^* \quad x_t^* \quad g_t \quad z_t \quad \tilde{\pi}_t \quad \tilde{x}_t \quad \tilde{L}_{t-1} \quad S_{t-1} \quad C_{t-1} \quad \tilde{\pi}_{t-1} \quad \tilde{x}_{t-1} \Big)'.$$

The corresponding matrices of the state space model are

$$\mathbf{C} = \begin{pmatrix}
1 & \theta_s(1) & \theta_c(1) & 1 & 0 & 4 & 1 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \mathbf{0_{K\times 7}} & & & \\
1 & \theta_s(\tau_K) & \theta_c(\tau_K) & 1 & 0 & 4 & 1 & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & \mathbf{C}_1\mathbf{F}^4 & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$\mathbf{F} = \begin{pmatrix}
a_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & a_{11} & a_{12} & 0 & 0 & 0 & 0 & a_{13} & a_{14} & & & \mathbf{0_{3\times 5}} & & \\
0 & a_{21} & a_{22} & 0 & 0 & 0 & 0 & a_{23} & a_{24} & & & & & \\
 & & & 1 & 0 & 0 & 0 & & & & & & & \\
 & \mathbf{0_{4\times 3}} & & 0 & 1 & 1 & 0 & & & \mathbf{0_{4\times 7}} & & & & \\
 & & & 0 & 0 & 1 & 0 & & & & & & & \\
 & & & 0 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & 0 & 0 & 0 & 0 & 0 \\
\frac{a_3}{2} & \frac{a_3}{2}\theta_s(1) & \frac{a_3}{2}\theta_c(1) & 0 & 0 & 0 & 0 & -\frac{a_3 b_1}{2} & a_1 - \frac{a_3 b_2}{2} & \frac{a_3}{2} & \frac{a_3}{2}\theta_s(1) & \frac{a_3}{2}\theta_c(1) & -\frac{a_3 b_1}{2} & a_2 - \frac{a_3 b_2}{2} \\
 & & & & & & & & & & & & & \\
 & \mathbf{I_{3\times 3}} & & & & \mathbf{0_{3\times 6}} & & & & & & & & \\
 & & & & & & & & & & & \mathbf{0_{5\times 5}} & & \\
 & & & \mathbf{0_{2\times 7}} & & & & \mathbf{I_{2\times 2}} & & & & & & \\
 & & & & & & & & & & & & &
\end{pmatrix},$$

where $\mathbf{C}_1$ denotes the first row of $\mathbf{C}$. The matrices $\mathbf{D}$ and $\mathbf{G}$ are assumed to be diagonal with standard deviations of state and measurement innovations on their diagonal. Lastly, noting equations (27) and (28) the column vectors for the constants γ and μ are given by

$$\gamma \quad = \quad \begin{pmatrix} \mathcal{A}(\tau_1) & \dots & \mathcal{A}(\tau_K) & 0 & 0 & \gamma^{shsr} & 0 & \gamma^{lhlr} \end{pmatrix}', \tag{29}$$

where $\gamma^{shsr} = \mathcal{A}(\tau_1) + \mathbf{C}_1(\mathbf{I} + \mathbf{F} + \mathbf{F}^2 + \mathbf{F}^3)\mu$ and $\gamma^{lhlr} = \mathcal{A}(40) + \theta_s(40)\bar{S} + \theta_c(40)\bar{C}$ and

$$\mu = \begin{pmatrix} 0 & a_{10} & a_{20} & 0 & 0 & 0 & 0 & 0 & -a_3[\theta_s(1)\bar{S} + \theta_c(1)\bar{C}] & 0 & 0 & 0 & 0 & 0 \end{pmatrix}',$$

respectively.

B Parameter restrictions to rule out arbitrage in the dynamic Nelson-Siegel model

In this Annex we explain the no-arbitrage adjustment term $\mathcal{A}(\tau)$ in the yield equation (8). As shown by Christensen et al. (2011) and, in a discrete-time setting, Li et al. (2012), pricing bonds under a specific choice of risk-neutral factor dynamics renders the joint dynamics of bond yields arbitrage-free, gives rise to factor loadings having the Nelson-Siegel functional form, but implies an additional intercept term that is not present in the standard – statistically motivated – Nelson-Siegel formulation.

Starting from the definition of the state variable ξ_t as in Annex A, we define a factor vector $F_t = [L_t, \bar{\xi}_t]$, where $\bar{\xi}_t$ equals our state vector ξ_t except that the first three elements are re-shuffled so that $\tilde{L}$ appears after the slope and curvature factor S and C. The so-constructed factor vector F_t has the three Nelson-Siegel factors L_t, S_t and C_t lining up upfront. Note further that L results as a linear combination of the states $\tilde{L}$, g, z and π^*.[19] We further group $F_t = [F_t^u F_t^o]$ with $F_t^u = [L_t, S_t, C_t]$ and F_t^o capturing the rest of the variables. Based on that partitioning of factors we represent the short-rate equation as

$$i_t = \delta_0 + \delta_u' F_t^u + \delta_m' F_t^m = \delta_0 + \delta' F_t$$

with obvious notation. Let $P_t(\tau)$ denote the time-t price of a zero-coupon bond with residual maturity τ. If there are risk-neutral factor dynamics (labelled by $\mathbb{Q}$)

$$F_t = c^{\mathbb{Q}} + \Phi^{\mathbb{Q}} F_{t-1} + v_t^{\mathbb{Q}}, \quad v_t^{\mathbb{Q}} \sim \mathcal{N}(0, \Omega) \tag{30}$$

[19] As $L_t = \tilde{L}_t + i_t^* = \tilde{L}_t + 4g_t + z_t + \pi_t^*$.

so that bond prices satisfy

$$P_t(\tau) = e^{-i_t} E_t^{Q} P_{t+1}(\tau - 1), \quad P_t(0) = 1,$$

then the joint evolution of bond prices is arbitrage-free. Moreover, the solution to the pricing equation is exponentially affine in factors

$$P_t(\tau) = \exp\big(a(\tau) + b(\tau)' F_t\big)$$

where coefficients $a(\tau)$ and $b(\tau)$ satisfy the well-known difference equations

$$\begin{aligned} a(\tau+1) &= a(\tau) + b(\tau)' c^{\mathbb{Q}} + \frac{1}{2} b(\tau)' \Omega b(\tau) - \delta_0 \\ b(\tau+1)' &= b(\tau)' \Phi^{\mathbb{Q}} - \delta', \end{aligned}$$

with $a(1) = -\delta_0$ and $b(1) = -\delta$. Moreover, as shown by Li et al. (2012), if $\Phi^{\mathbb{Q}}$ is of the form

$$\Phi^{\mathbb{Q}} = \begin{pmatrix} \Phi^{\mathbb{Q}}_{uu} & 0 \\ \Phi^{\mathbb{Q}}_{mu} & \Phi^{\mathbb{Q}}_{mm} \end{pmatrix}, \quad \Phi^{\mathbb{Q}}_{uu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & \lambda e^{-\lambda} \\ 0 & 0 & e^{-\lambda} \end{pmatrix},$$

then $b(\tau)$ exhibits the specific Nelson-Siegel loadings (in price space) for the first three factors L, S and C, and zero on the other factors,

$$b(\tau) = \Big[-n, -\frac{1-e^{-\lambda n}}{\lambda}, n e^{-\lambda n} - \frac{1-e^{-\lambda n}}{\lambda}, 0, \ldots, 0 \Big]'.$$

In addition, the zero restrictions on $\Phi^{\mathbb{Q}}$ imply that the expression for $a(\tau)$ simplifies to

$$a(\tau+1) = a(\tau) + b(\tau)' c^{\mathbb{Q}} + \frac{1}{2} b_u(\tau)' \Omega_{uu} b_u(\tau) - \delta_0, \tag{31}$$

where $b_u(\tau)$ contains the first three elements of $b(\tau)$ and Ω_{uu} is the upper 3-by-3 block of Ω.

Recalling that $F_t = [L_t, \bar{\xi}_t]$ is just an extension of our state vector ξ_t, the transition equation for F_t is readily derived from that of ξ_t described in Annex A. It is affine, as the stipulated (unobserved) risk-neutral dynamics in (30) above, but depends on the physical (no $\mathbb{Q}$ label) parameters:

$$F_t = c + \Phi F_{t-1} + v_t, \quad v_t \sim \mathcal{N}(0, \Omega)$$

The variance-covariance matrix Ω of state innovations is the same under both the risk-neutral and the physical measure. For our factor vector $F_t = [L_t, \xi_t]$ it follows from the dynamics of ξ_t and the link of L_t to $\tilde{L}$, z_t, g_t and π_t^* that Ω_{uu} in (31) is given by

$$\Omega_{uu} = diag(\sigma_{\tilde{L}}^2 + \sigma_{\pi^*}^2 + 16\sigma_g^2 + \sigma_z^2, \sigma_s^2, \sigma_c^2),$$

where σ_i^2 denotes the variance of the innovation ε_t^i of variable i in our model. Parameters governing the risk-neutral and physical dynamics are linked as

$$c^{\mathbb{Q}} = c - \Omega^{\frac{1}{2}}\lambda_0, \quad \Phi^{\mathbb{Q}} = \Phi - \Omega^{\frac{1}{2}}\Lambda$$

where λ_0 and Λ ('market prices of risk') are a vector and a matrix, respectively, of appropriate dimension.

Mapping bond prices into yields using $y_t(\tau) = -\frac{1}{\tau}\ln P_t(\tau)$, we have

$$y_t(\tau) = \mathcal{A}(\tau) + \mathcal{B}(\tau)'F_t$$

where $\mathcal{A}(\tau) = -\frac{1}{\tau}a(\tau)$ and $\mathcal{B}(\tau) = -\frac{1}{\tau}b(\tau)$. That is, $\mathcal{B}(\tau)$ has now the Nelson-Siegel loadings for bond yields as the first three entries, and $\mathcal{A}(\tau)$ is the intercept appearing in (8).

The risk-neutral dynamics and cross-sectional pricing equations are parsimoniously parameterized. The Nelson-Siegel tuning parameter λ is calibrated as described in the main text. The relevant variance-covariance matrix Ω_{uu} is implied by the time series estimates under the physical measure as explained above. As we are working with latent factors, the parameter δ_0 in the short-rate equation is not identified and can be arbitrarily calibrated. While it is common to set it to zero, we choose to set $\delta_0 = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ so that (as $a(1) = -\delta_0$) $\mathcal{A}(1) = -a(1) = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ as specified in the main text. Finally, we set the risk-neutral VAR intercept $c^{\mathbb{Q}}$ equal to zero. This is a somewhat ad-hoc choice to prevent additional parameters to enter our setup and is tantamount to imposing a restriction on the market price of risk vector λ_0, given the estimates of c and Ω of the physical dynamics. While under that specific choice of $c^{\mathbb{Q}}$ model-implied bond yield dynamics are arbitrage-free, it is eventually an empirical question, whether $c^{\mathbb{Q}} = 0$ is an overly restrictive assumption. Via its impact on $\mathcal{A}(\tau)$, the choice of $c^{\mathbb{Q}}$ affects the (average) slope of the yield curve as argued in the main text. It turns out empirically that the model fits the average slope in the data fairly well so that the parameter restriction appears non-problematic from this perspective.[20]

[20] The mean absolute fitting errors for the 2-, 5- and 10-year maturities are, respectively, 5bps, 9bps and 12bps for the US and 8bps, 10bps and 7bps for the euro area.

C Data

The following table provides an overview of the quarterly data used in this study. For the United States, inflation and GDP data are taken from the FRED-database of the Federal Reserve Bank of St. Louis and yields from Gürkaynak et al. (2007). Sources for euro area data are the ECB's Statistical Data Warehouse, Deutsche Bundesbank, Bloomberg, and Consensus Economics. `ACRONYMS` refer to codes in the respective databases. Synthetic, pre-1999 euro area data are in fixed composition of member countries (except for HICP which is in full composition for completing the series over the sample period starting 1995-97). The overall sample period covers 1961 Q2–2019 Q4 for the United States and 1995 Q1–2019 Q4 for the euro area.

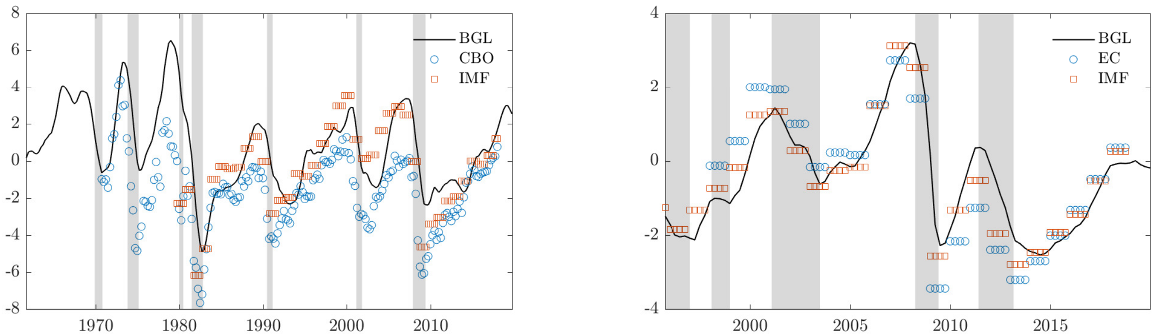
Table 2: Data used in this study

Variable	**US**	**EA**
GDP	`GDPC1`	`MNA.Q.Y.I8.W2.S1.S1.B.B1GQ._Z._Z._Z.EUR.LR.N`
Consumer Prices, all items	`CPIAUCSL` which is seasonally adjusted	HICP – `ICP.M.U2.N.000000.4.INX` seasonally adjusted using X-12-ARIMA for data 1995 Q1–1997 Q1, subsequently seasonally adjusted series `ICP.M.U2.Y.000000.3.INX`.
Quarter-end zero-coupon yields	Data by Gürkaynak et al. (2007)	Zero-coupon yields on German government bonds up to 2005 Q4, subsequently midquotes from OIS bid and ask: `FM.B.U2.EUR.RT.SI.EUREON3M_.ask` or `FM.B.U2.EUR.RT.SI.EUREON3M_.bid`, etc. (Sources: Deutsche Bundesbank, ECB Statistical Data Warehouse) .
Short-horizon short-term interest rates expectations	Consensus Economics forecasts of the 3-months T-Bill, 1-year ahead (as of 1989 Q2)	Consensus Economics forecasts of the 3-months Euribor, 1-year ahead (as of 1995 Q2)
Long-horizon long-term interest rates expectations	Consensus Economics forecasts of the 10-year Treasury bond, 6-10 years ahead (as of 1989 Q2)	Consensus Economics forecasts of the 10-year German bund, 6-10 years ahead (as of 1995 Q2)

D Comparison with institutional output gap estimates

Figure 13 plots model-specific output gap estimates against institutional ones. Generally, the model-specific estimates co-move with institutional ones and, by and large, there is a high degree of consistency in the timing of business cycle turning points. While our model-based estimate for the United States lies mostly between the institutional estimates from the IMF and CBO, slack in the aftermath of the Global Financial crisis is more swiftly absorbed in our model-based estimate of our benchmark model than in the official estimates. In contrast, adding long-horizon long term interest rate expectations as an additionally observable to inform the model about the low frequency component of yields seems to negatively affect the output gap estimate. For the euro area, our model-based output gap estimates closely follow those estimated by the IMF or the European Commission.

Figure 13: Output gaps compared to official estimates



Note: The left panel shows institutional output gap measures for the United States from the Congressional Budget Office (CBO) and the IMF against our model-based estimates. NBER recessions in gray. The right panel shows institutional output gap measures for the euro area from the European Commission (EC) and the IMF against our model-based estimates. CEPR recessions in gray.

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