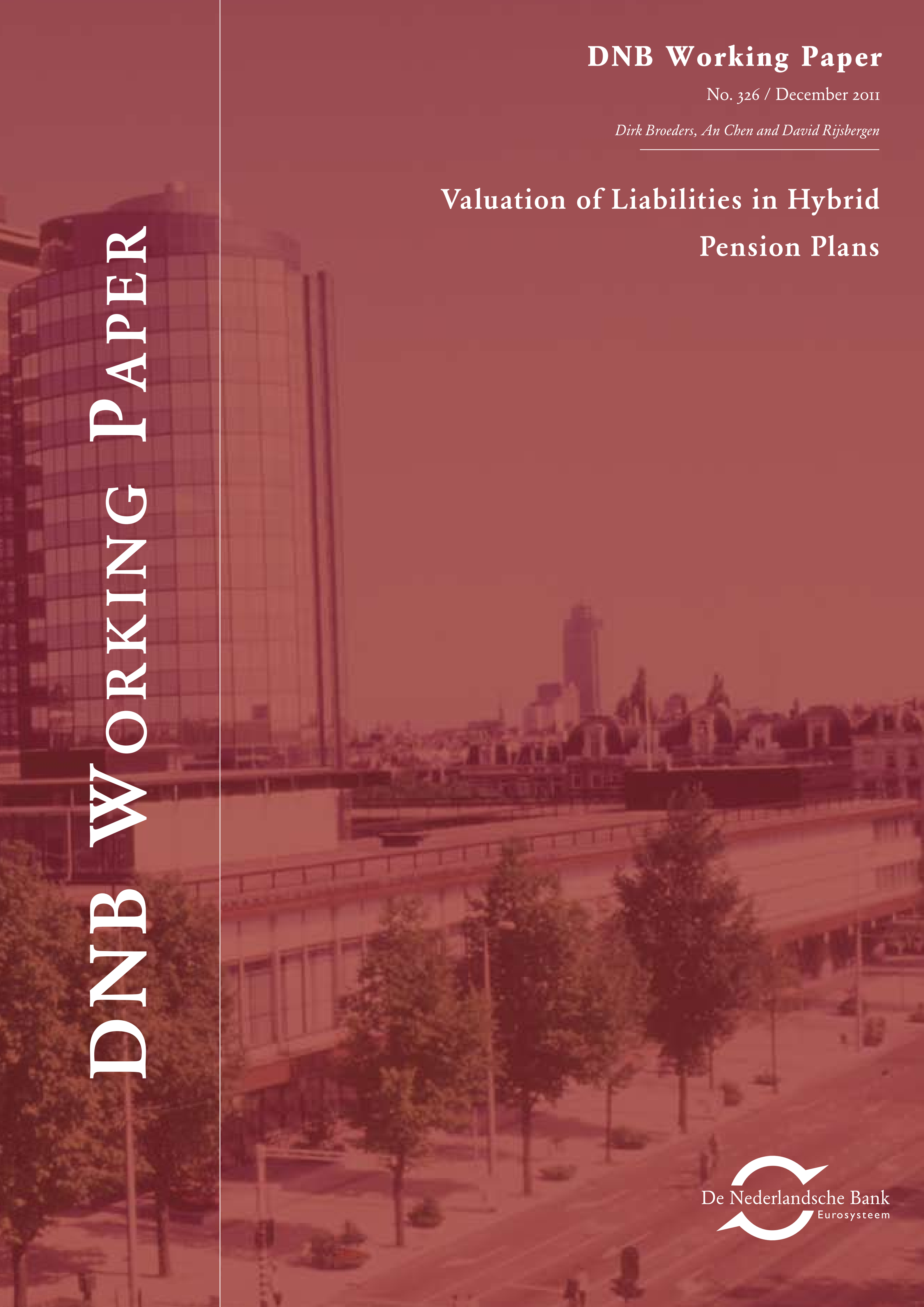

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* Views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.

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VALUATION OF LIABILITIES IN HYBRID PENSION PLANS

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Abstract. In this paper we derive an analytic valuation formula for a generalized form of liabilities in hybrid pension plans taking account of both equity and interest rate risk. Comparative statistics are carried out to show the relevance of some key parameters in defining the hybrid pension plans, particularly the indicator of hybridity and the equity allocation in the pension fund's investment policy. We find that both the level of hybridity and the equity allocation of the pension fund impact the value of hybrid plan liabilities. This should affect the negotiation between employers and employees on total labor compensation.

Keywords: *Market consistent valuation, overlapping generations, forward risk adjusted measure, Vasiček*

JEL-Codes: *G12, G13, G23*

1. Introduction

The recent financial crisis has fuelled the interest in market consistent valuation of pension liabilities. Traditionally, a pension plan could be classified either as a defined benefit (DB) scheme or a defined contribution (DC) scheme. Valuation of liabilities in these types

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of pension plans is relatively straightforward. The international structure of occupational pension plans, however, has significantly changed during the last decade owing to the development of hybrid pension plans that combine elements of traditional DB and DC plans. Although hybrid pension plans were introduced in the United States back in the 1980s, their popularity has soared over the last ten years as an innovative way of evenly allocating funding risks among stakeholders. Between 2001 and 2008 the number of hybrid pension plans in the U.S. more than doubled to approximately 2,900 plans, which account for over 30 percent of corporate pension plan participants (PBGC, 2010).

Not surprisingly, in recent years the conversion to hybrid pension plans has received significant academic interest. This paper aims to contribute to existing research by being, to the best of our knowledge, the first to rigorously analyze the market consistent valuation of liabilities in hybrid pension plans. We develop a model which allows for a continuum of pension plans, ranging from pure defined benefit to pure defined contribution. Before discussing the valuation framework we will briefly discuss hybrid pension schemes.

Although hybrid pension plans come in different forms, they all effectively involve a degree of risk sharing between the key stakeholders, being employers and employees. The distribution of risk factors over the stakeholders differs for traditional defined benefit plans, defined contribution plans and hybrid pension plans. In general, most of the risks in a traditional DB plan lie with the plan sponsor because the sponsor is ultimately responsible for meeting the pension obligations. By contrast, the bulk of risks in a pure DC plan are borne by the plan members themselves. In hybrid pension plans the risks are generally shared. Members in a hybrid pension plan typically face investment risk and conversion risk when they retire as well as wage growth or inflation risk.

The most common forms of hybrid plans in the U.S. so far are cash balance plans and pension equity plans. Both plans have DC elements as benefits are paid out in lump sum rather than as a life annuity. Participants can also generally take a pre-retirement lump sum from the plan if they leave their employer before they retire. Both plans are, however, legally classified as DB schemes since they accrue benefits to employees under a fixed formula. In a cash balance plan for instance, a percentage of pay (salary credits) plus a guaranteed rate of return (interest credits) are periodically added to the member's account. The latter is typically tied to the yield on long-term government bonds. So contrary to a pure DC plan, the cash balance account grows at a rate that is not directly related to the plan's actual investment earnings, but is pre-determined and specified in the plan document. If the specified plan rate exceeds the return on the plan's assets, then a deficit will occur for which the sponsor is liable. From the perspective of the sponsor, cash balance plans therefore operate somewhat similar to a traditional DB scheme (Niehaus and Yu, 2005). Another prominent example of hybrid pension plans are career average defined benefit plans with contingent indexation in countries such as Canada and the Netherlands, in which the level of indexation is contingent on the pension fund's solvency status. These plans have the feature of a DB plan because the yearly accrual of pension rights is specified in a similar manner as in a traditional DB plan. However, these plans also have DC characteristics as their indexation level is related to the fund's solvency status and therefore contingent on the actual investment returns. If their contribution rate is also fixed for a prolonged period of time, these plans are known as collective defined contribution schemes (Ponds and Van Riel, 2007).

Hybrid pension plans have been embraced in an attempt to mitigate the drawbacks inherent in pure DC and DB plans. The most common motivating factor from the sponsor's perspective is reducing the volatility of plan contributions. In a traditional DB scheme, the sponsor promises the plan beneficiaries a final level of pension benefits, which are generally paid out as a life annuity. As such, the plan's sponsor is exposed to substantial investment and longevity risks which could lead to volatile and unplanned contributions. Liabilities in hybrid pension plans, however, are typically less sensitive to interest rate fluctuations than liabilities in traditional DB schemes. An interest rate decline, for instance, increases the discounted plan liabilities in a cash balance plan, but also reduces the plan's benefits due to lower interest credits. These opposite forces partially offset each other (Hill, Pang and Warshawsky, 2010). Hence, hybrid pension plans tend to reduce the volatility of the sponsor's contributions.

The changing workforce demographics are an additional motivation for the global transition towards hybrid pension plans, as these plans are better suited for aging populations, which necessitate longer working lives and more flexible labor markets. Employees in hybrid pension plans generally accrue benefits more evenly over their years of service than those in traditional DB plans. This makes it easier for them to switch jobs without losing out on their pension rights. For instance, Coronado and Copeland (2004) find evidence that conversions to cash balance plans are partly influenced by labor market conditions. They find that industries with younger, more mobile workers and tighter labor markets tend to show more conversions to hybrid plans. As such, hybrid pension plans stimulate dynamic and mobile workforces. Furthermore, hybrid plans typically lack the early retirement incentives inherent to many DB plans and do not penalize employees who continue working after their retirement age (Johnson and Steuerle, 2004). This is supported by Friedberg and Webb (2003), who find that workers in DC plans, which typically have similar retirement incentives as hybrid plans, retire approximately two years later on average than their counterparts in traditional DB plans.

Hybrid pension plans also offer attractive features from the perspective of employees. Cash balance plans are typically easier for employees appreciate than traditional DB plans. For example, employees can easily add the annual pay and interest credits to their lump sum account to understand how the value of their pension is expected to grow over time. By contrast, employees often do not know how to determine the value of a deferred annuity that traditional DB plans offer. Moreover, hybrid pension plans provide more portable pension benefits than traditional DB plans, which is attractive to young and mobile workers. This is supported by Clark and Schieber (2004) who find that the majority of workers who leave their employer (voluntary or involuntary) before the age of 55 can expect to receive larger pension benefits in hybrid pension plans than in traditional DB plans.

Hybrid pension plans also provide certain advantages over pure DC plans. In a pure defined contribution scheme employers are only obliged to make fixed contributions. The employees thus bear the brunt of the risk and typically face uncertain replacement rates caused by fluctuations in the capital markets (Bodie and Merton, 1992). Hybrid pension plans, however, reduce the variability in the range of potential outcomes as they may include a DB element such as guaranteed benefits or returns. Another feature of a pure DC plan is that plan members have extensive control over their account's investment strategy, which theoretically enables them to shape their portfolio to their individual risk and return preferences. In practice, however, many plan members are not willing or able to determine a suitable investment strategy and are thus prone to making poor decisions. Hybrid pension plans are managed by dedicated trustees and professionals who protect employees against these pitfalls. Finally, hybrid pension plans also appear to have a tax advantage over pure DC plans. Niehaus and Yu (2005), for instance, find empirical evidence in support of the excise tax avoidance hypothesis in the United States. According to this hypothesis, several companies have transformed their DB plans into cash balance plans as conversion allows them to avoid paying taxes on their excess pension assets. If they had converted to a pure DC plan instead, they would have lost a substantial part of their excess assets to stiff excise duties. However, tax avoidance does not come free as cash balance plans typically carry higher administrative costs than pure DC plans (Niehaus and Yu, 2005).

Despite the advantages of hybrid plans, the growing number of companies converting their DB plans to hybrid plans has also generated some controversy. Hybrids are far more difficult to comprehend compared to defined benefit contracts. Some view hybrid plans as attempts by employers to reduce pension costs. Academic evidence, however, suggests that cost reduction is not the main rationale behind conversion (see Coronado and Copeland, 2004, and Mitchell and Mulvey, 2004). Others perceive hybrid plans as age discriminatory, because older workers have fewer years to compound interest credits than younger ones. Mitchell and Mulvey (2004) interestingly point out that this concern is typically not raised in a DC context, even though the benefit accrual patterns in a DC plan are almost identical to those in hybrid plans. Moreover, most court cases have rejected claims of inherent age discrimination, and in the Pension Protection Act (PPA) U.S. Congress declared that cash balance plans and pension equity plans that satisfy certain safe-harbor requirements will not be viewed as age discriminatory (Hill, Pang, and Warshawsky, 2010). These legislative developments might further stimulate the conversion to hybrid pension plans in the near future.

The paper focuses on the market consistent valuation of hybrid pension liabilities. Valuation and the risk factors that drive valuation are not only important to beneficiaries in decision making on optimal life cycle saving and investing, but also to pension funds to perform optimal risk management, and for sponsors to manage labor costs. Furthermore, a market consistent approach to valuation promotes economic welfare as it allows stakeholders to make optimal financial decisions. The valuation of liabilities in hybrid pension plans, on the other hand, is also more complex than for traditional DB or DC plans as the value of liabilities depends on specific hybrid plan features, such as the periodic interest crediting rate. This rate can be fixed, variable or a combination of the two and is typically determined by the employer.

The remainder of this paper is organized as follows. Section 2 briefly reviews the concept of market consistent valuation of pension liabilities. In Section 3 we develop our model followed by the actual valuation in Section 4. Section 5 provides numerical results and shows the impact of key parameters on the outstanding liability over time. Our conclusions are set out in the final section.

2. Valuation of pension liabilities

We start our endeavor with the valuation of assets. Asset values are typically based on observed market prices in public markets. If prices cannot be directly observed in the market, they can be replicated through similar instruments. Hence, the price of the asset will correspond to the observed market price of comparable instruments. If neither case applies, it is common practice to use a market consistent valuation model based on generally accepted principles such as the no-arbitrage condition. Financial economics promotes market consistent valuation of liabilities (Exley, Mehta and Smith, 1997). Although there appears to be no deep and liquid market for pension liabilities, market consistent valuation is still possible on the basis of the replication principle. The key assumption here is that in a world without arbitrage opportunities, the market consistent value of a pension liability equals the market price of the investment strategy that generates exactly the required cash flows under all future states of the world.

The replicating investment strategy for guaranteed nominal or indexed pension liabilities consists of nominal or inflation linked bonds and other fixed income securities with negligible default risk, such as interest rate swaps. Defined benefit liabilities may be valued at the market nominal or real term structure of interest rates. In case of pure DC liabilities the value of liabilities, by definition, equals the value of the assets. Recently, the valuation of so called contingent pension liabilities has received significant attention. Nijman and Koijen (2006) apply pricing kernels to value conditionally indexed pension liabilities, while De Jong (2008) employs models for asset pricing in incomplete markets to value pension liabilities that carry unhedgeable wage-indexation risks. Broeders (2010) specifically takes into account sponsor risk to value contingent pension liabilities where the pension plan sponsor underwrites the defined benefits. Moreover, Broeders and Chen (2010) analyze the market consistent valuation of pension liabilities in a contingent claim framework using barrier options to replicate the possibility of an early regulatory closure of the pension plan. This approach of market consistent valuation is also promoted for insurance contracts see, e.g., Grosen and Jørgensen (2002) and Ballotta, Haberman and Wang (2006)

For hybrid pension liabilities the market consistent valuation may require more assumptions than the valuation of pure DB or DC obligations. The value of hybrid liabilities may, for instance, also require assumptions on the volatility of financial markets. The approach that we follow below is largely similar to that for the valuation of contingent pension liabilities. If pension benefits are a function of some underlying stochastic process, e.g. the funding ratio or asset returns, their value can be derived using modern valuation techniques for derivatives.

3. Model setup

Our model is based on a pension consisting of 55 overlapping generations, ranging from age 25 to 80. The 55 cohorts are homogenous, particularly with respect to population size and preferences. We assume that all individuals start working at age 25 and retire at the age of 65, at which they have a life expectancy of 15 years.

We consider a run off scenario, which implies that no new participants can join the fund after $t = 0$. This means that we consider a pension fund operating for 55 periods. At time 0, we have 40 working generations and 15 retired generations. In each period, one generation dies out. Until $t = 40$, one more generation will retire in each period. Hence, after $t = 40$ only retired generations are left. We assume that demographic risk is absent.

3.1. **Outstanding liability.** In order to record the outstanding liability of the pension fund at time t, we need to distinguish between two cases: $t \leq 40$ (when there are still working generations) and $t > 40$ (when there are only retired generations). The key variable in the model is the discounted outstanding liability at time t, which is expressed as

$$L_t^T = \begin{cases} \sum_{x=t}^{40} \left(\sum_{i=41}^{55} e^{-\int_t^i r_u du} Z_i^x - \sum_{i=x}^{40} e^{-\int_t^i r_u du} P_i^x \right) + \sum_{x=41}^{55} \sum_{i=x}^{55} e^{-\int_t^i r_u du} Z_i^x, & t \leq 40 \\ \sum_{x=t}^{55} \sum_{i=x}^{55} e^{-\int_t^i r_u du} Z_i^x, & t > 40 \end{cases}$$

where we use Z_i^x to denote the periodic pension benefit payments made to generation x at time i, which will be specified further. All cash flows are discounted to time t with the discount factor $e^{-\int_t^i r_u du}$. Here we allow the interest rate to be stochastic. Since for $t \leq 40$, we have both working and retired generations in the model, the working generations pay contributions to the pension fund. P_i^x is the periodic contribution working generation x pays at time i, and is assumed to be deterministic. It is not unusual in practice that the contribution rate is fixed, e.g. by using a fixed discount rate. In return the members receive an entitlement to future pension payments. Hence, their target liability is the difference in brackets in the formula. The retired generations receive annual pension payments only. As time goes by, the population of the pension fund in our model will consist of fewer and fewer working generations until the entire population consists of retired generations. Finally at $t = 55$ the last generation passes away and the pension fund is closed.

3.2. **Benefit adjustment mechanism.** In the interest of clarity, we assume that all retired generations initially receive similar benefits at time i, so that $Z_i^x = Z_i$. These benefits, however, can be adjusted over time. The generation retiring in 20 years from now will therefore have different benefit levels than the current retirees. In the following, we consider two different benefit adjustment mechanisms.

- In the first mechanism, the benefit payment depends on the **cumulative return** from the evaluation point until the actual payment. The benefit at time i is defined by

$$Z_i^I = Z \exp\{y_i - y_t\} \quad \text{where} \quad y_u = \alpha r_u^p + (1-\alpha) R_u,$$

$$\text{with } r_u^p = \ln \frac{X_u}{X_0} \qquad \text{and} \qquad R_u = \int_0^u r_u du$$

where Z is a constant. In this definition r_u^p is the cumulative log return of the pension fund's asset X over $[0, u]$ which will be specified below. R_u is the cumulative interest rate. Time t is the valuation time of the liability. The parameter $\alpha \in [0, 1]$ is an indicator of the degree of "hybridity". The adjustment mechanism shows that the ultimate benefit is a linear combination of the log-return on the pension fund's assets $r_i^p - r_t^p$ and the log-return on a default free bond $R_i - R_t$.

- In the second mechanism, the benefit payment at time i is coupled with the **periodic increments** in assets and the market interest rate. The benefit at time i in this case is expressed as

$$Z_i^{II} = Z \exp\{y_i - y_{i-1}\}$$

  in which y_u is defined as in the first scheme. The exclusive difference in this scheme lies in the fact that the benefit is dependent on the asset and interest rate movements between the time period $[i-1, i]$.

Based on these two schemes, we can define a continuum of pension contracts through the parameter α:

- for $\alpha = 0$, the pension contract has characteristics of a defined benefit plan as the performance for the beneficiary is the risk-free return. However, the pension fund might still invest in equities, thereby creating risk for the beneficiary.
- for $\alpha = 1$, the pension contract has characteristics of a defined contribution plan as the beneficiary receives the actual return on the pension fund's assets. However, the pension fund might still invest in bonds, thereby creating certainty for the beneficiary.
- for $\alpha \in (0, 1)$, the pension contract is a hybrid plan consisting of DB and DC elements.

We can rewrite the periodic pension benefit Z_i at time i depending on the development of the asset and the interest rate throughout the period $[t, i]$ in scheme I and the development in the period $[i-1,i]$ in scheme II as follows:

$$Z_i^I = Z \cdot \left(\frac{X_i}{X_t}\right)^\alpha \exp\left\{(1-\alpha)\int_t^i r_u du\right\}$$
$$Z_i^{II} = Z \cdot \left(\frac{X_i}{X_{i-1}}\right)^\alpha \exp\left\{(1-\alpha)\int_{i-1}^i r_u du\right\}.$$

This completes the definition of the benefit structure.

3.3. **Available assets and investment strategy.** Next we describe the pension fund's assets and investment strategy. We assume that the pension fund operates in a financial market with two traded assets over the life time of the fund $[0,T]$: a diversified equity portfolio S and a risk-free bond B. Risk-free in this context refers to the absence of default risk. Given the risk-neutral pricing measure P, we can assume that the value of the equity portfolio and that of the risk-free bond evolve as follows:

$$dS_t = r_t S_t dt + \sigma_S S_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2)$$
$$dB_t = r_t B_t dt$$
$$dr(t) = (b - ar(t))\, dt + \sigma_r\, dW_t^1.$$

in which W_t^1 and W_t^2 are two independent one-dimensional Brownian motions and σ_S is the volatility of the equity portfolio. The factor ρ represents the correlation between the stock and the interest rate market. A positive correlation means that an increase in interest rates is likely to coincide with positive equity returns. We allow for stochastic interest rates and the term structure of interest rates is modeled using the Vasiček (1977) model, in which a is the speed factor, $\frac{b}{a}$ is the long-term mean interest rate and σ_r is the volatility of the short-term rate. These parameters are all constant.

In the Vasicěk model, the cumulative interest rate under the risk neutral measure P is expressed as

$$R_s = \int_0^s r(u)du = \mathcal{D}(0,s)r(0) + b\int_0^s \mathcal{D}(u,s)\, du + \sigma_r \int_0^s \mathcal{D}(u,s) dW_u^1$$
$$\Rightarrow R_s - R_t = \int_t^s r(u)du = \mathcal{D}(t,s)r(t) + b\int_t^s \mathcal{D}(u,s)\, du + \sigma_r \int_t^s \mathcal{D}(u,s) dW_u^1$$

in which $\mathcal{D}(t,s) := e^{at}\int_t^s e^{-au}\,du$.

Suppose the pension fund invests a fixed percentage β in the equity portfolio and, consequently, $1-\beta$ in risk free bonds. Over time the fund follows a continuous rebalancing strategy to keep the asset allocation stable. Leaving pension benefit payments and premium incomes aside, the pension fund's assets change as follows

$$\begin{aligned} dX_t =& X_t\left(\beta\frac{dS_t}{S_t} + (1-\beta)\frac{dB_t}{B_t}\right) \\ =& r_t X_t dt \ + X_t\beta\sigma_S(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \end{aligned} \tag{1}$$

given the risk-neutral measure P. The cumulative log-return of X_t over $[0,t]$ is expressed as

$$\begin{aligned} r_t^p =& \ln(X_t/X_0) = \int_0^t \left(r_u - \frac{1}{2}\beta^2\sigma_S^2\right)du + \int_0^t \beta\sigma_S\left(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2\right) \\ \Rightarrow\ r_s^p - r_t^p =& \int_t^s \left(r_u - \frac{1}{2}\beta^2\sigma_S^2\right)du + \int_t^s \beta\sigma_S\left(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2\right). \end{aligned} \tag{2}$$

This completes the definition of the assets and investment strategy.

4. Valuation of the pension liabilities

We now turn our attention to the valuation of liabilities in hybrid pension plans. We value the liabilities for both schemes described in Section 3. The development of the pension benefits in these schemes is a function of the performance of the pension fund's assets and the risk-free rate. Our challenge is to determine the time-t value of the outstanding liability for $t = 1,\cdots,55$. The market consistent value of the liability at time t is expressed as[1]

$$E[L_t^T|\mathcal{F}_t] = \begin{cases} \sum\limits_{x=t}^{40}\left(\sum\limits_{i=41}^{55} E[e^{-\int_t^i r_u du}Z_i^x|\mathcal{F}_t] - \sum\limits_{i=x}^{40} E[e^{-\int_t^i r_u du}P_i^x|\mathcal{F}_t]\right) + \sum\limits_{x=41}^{55}\sum\limits_{i=x}^{55} E[e^{-\int_t^i r_u du}Z_i^x|\mathcal{F}_t], & t \le 40 \\ \sum\limits_{x=t}^{55}\sum\limits_{i=x}^{55} E[e^{-\int_t^i r_u du}Z_i^x|\mathcal{F}_t], & t > 40 \end{cases} \tag{3}$$

[1] Since we have used L_t^T to denote the discounted total liability, the time-t market consistent value is simply the expected value of L_t^T conditional on the information structure $\mathcal{F}_t$.

Given the run off scenario it is necessary to distinguish between periods when there are both active and passive members in the pension fund and periods during which there are only retirees left. This is based on the principle that the expectation of a sum is the sum of expectations. Since the premium payments P_i^x are assumed to be deterministic, the calculation boils down to determining the expectation $E[e^{-\int_t^i r_u du} Z_i^x | \mathcal{F}_t]$, where Z_i^x takes different values in the first and second benefit mechanisms. The time-t market consistent value of the total outstanding liability $E[L_t^T | \mathcal{F}_t]$ is then determined by substituting this expected value.

4.1. **Market value of liability under benefit scheme I.** In the first benefit mechanism, benefits depend on the cumulative return from t to i:

$$\begin{aligned}\bar{Z}_{t,i}^I :=& E[e^{-\int_t^i r_u du} Z_i^I | \mathcal{F}_t] \\ =& Z \cdot E\left[e^{-\int_t^i r_u du} \exp\{\alpha(r_i^p - r_t^p) + (1-\alpha)(R_i - R_t)\} \Big| \mathcal{F}_t\right]. \end{aligned} \tag{4}$$

Substituting the definition of the cumulative return (2) in (4), we obtain

$$\begin{aligned}\bar{Z}_{t,i}^I =& Z \cdot E\left[\exp\left\{-\alpha \int_t^i \frac{1}{2}\beta^2 \sigma_S^2 du + \alpha \int_t^i \beta \sigma_S (\rho dW_u^1 + \sqrt{1-\rho^2} dW_u^2)\right\} du \Big| \mathcal{F}_t\right] \\ =& Z \exp\left\{\frac{1}{2}\beta^2 \sigma_S^2 (\alpha^2 - \alpha)(i-t)\right\}. \end{aligned} \tag{5}$$

The valuation formula becomes straightforward as one term in the accumulation factor in the benefit is the accumulated interest rate from t to i and this term offsets the discount factor. The solution of the expectation in the equation above results from the fact that $\alpha \int_t^i \beta \sigma_s (\rho dW_u^1 + \sqrt{1-\rho^2} dW_u^2)$ is normally distributed with mean 0 and variance $\alpha^2 \beta^2 \sigma_S^2 (i-t)$. We would like to add the following observations regarding the valuation formula:

- The valuation of liabilities is independent of the expected return on the pension fund's assets. This is a key result of risk neutral valuation. Note that the pension benefits are contingent on the performance of the underlying stochastic asset processes and as such are effectively a derivative.

- The key parameters in the market value of the liabilities are: the indicator of hybridity (α), the equity allocation (β) in the investment strategy of the pension fund and the volatility of the pension fund's assets (σ_S).
- It proves that $\bar{Z}^I_{t,i}$ is a U-shaped function of hybridity parameter α as it decreases in α for $\alpha \in [0, 1/2]$ and increases in α for $\alpha \in [1/2, 1]$.[2] The maximum value of $\bar{Z}^I_t$ is achieved for $\alpha = 0$ or 1. For $\alpha = 0$ (or 1) the pension benefit is equivalent to a pure defined benefit scheme (or a pure defined contribution scheme). In both cases ($\alpha = 0$ or 1), the time-t value of a periodic benefit payment is given by $\bar{Z}^I_t \equiv Z$. If the benefit is a linear combination: $Z^I_i = Z\left(\alpha \exp\{r^p_i - r^p_t\} + (1-\alpha)\exp\{R_i - R_t\}\right)$, the present value of this payment is identical to Z. This holds because risk-neutral pricing is linear. But in our benefit formulation, the linear combination is for the exponent. The Jensen inequality therefore implies that the present value $\bar{Z}_{t,i}$ in this case is no greater than Z. Although $\bar{Z}_{t,i}$ is capped by Z, this does not imply that the realized benefit is always smaller than a pure defined benefit or a pure defined contribution plan. This depends on the realization of the returns from the equity and bond markets.
- The benefits $\bar{Z}^I_{t,i}$ decrease with the percentage of equity in the investment portfolio (β). This holds because for nontrivial values of $\alpha \in (0, 1)$, $\alpha^2 - \alpha$ is negative, which means that the increase in β leads to a decrease in the exponent. If the pension fund only invests in risk-free bonds, i.e. $\beta = 0$, the time-t value of the periodic pension liability is

  $$\bar{Z}^I_{t,i} \equiv Z.$$

  Interestingly, $\beta = 0$ or $\alpha = 0$ delivers the same market value, i.e. a constant Z.
- The valuation of liabilities does not depend on the initial value of the pension fund's assets X_0. What does matter, however, is the incremental rate of return on the risky asset and the stochastic interest rate. This follows directly from the

[2] Note that $\alpha^2 - \alpha = \left(\alpha - \frac{1}{2}\right)^2 - \frac{1}{4}$. For $\alpha \in [0, 1]$, the minimum value for $\alpha^2 - \alpha$ is obtained when $\alpha = \frac{1}{2}$ and the maximum is achieved for $\alpha = 0$ or 1.

benefit formulation. The periodic benefit adjustment depends on X_i/X_t, instead of X_i alone.

- Obviously, the liability increases in value when the promised periodic benefit payment Z is higher.

4.2. **Market value of liability under benefit scheme II.** In the second benefit mechanism, the market value of the outstanding liability $\bar{Z}^{II}_{t,i}$ depends on the period return and is expressed as

$$\begin{aligned}\bar{Z}^{II}_{t,i} :=& E[e^{-\int_t^i r_u du} Z_i^{II} | \mathcal{F}_t] \\ =& Z \cdot E\left[e^{-\int_t^i r_u du} \exp\{\alpha(r_i^p - r_{i-1}^p) + (1-\alpha)(R_i - R_{i-1})\} \Big| \mathcal{F}_t\right].\end{aligned}$$

In this case the accumulation factor and the discount factor do not cancel each other out, hence the derivation of $\bar{Z}^{II}_{t,i}$ becomes more complex. A practicable approach to valuing contingent claims is to transform the expectation from the risk-neutral probability measure P to a forward-risk-adjusted measure where zero coupon bonds are used as numeraires. The rationale behind this transformation is the numeraire-invariance principle (Geman, El Karoui and Rochet, 1995). In the Appendix, we express the expectation above under the forward risk-adjusted measure P^i in which the zero coupon bond with maturity date i is used as numeraire and obtain the following expression for the valuation of the pension liabilities under benefit mechanism II.

Proposition 4.1. *The time-t value $\bar{Z}^{II}_{t,i}$ is the product of three distinctive parts:*

$$\bar{Z}^{II}_{t,i} = \mathcal{M}(i-1,i) \cdot \mathcal{Q}(i-1,i) \cdot \mathcal{Y}(t,i-1)$$

in which

$$\begin{aligned}\mathcal{M}(i-1,i) =& Z \cdot \exp\Big\{ -\frac{\alpha}{2} \int\limits_{i-1}^{i} ((\eta^2(u,i) + \sigma_S^2 \beta^2(1-\rho^2)) - \gamma^2(u,i-1,i)) du\Big\} \\ & \cdot \exp\left\{(1-\alpha)\left(b \int_{i-1}^{i} \mathcal{D}(u,i)\, du + \sigma_r \int_{i-1}^{i} \mathcal{D}(u,i)\sigma(u,i) du\right)\right\}\end{aligned}$$

with

$$
\begin{aligned}
\mathcal{D}(t,s) :=& e^{at}\int_t^s e^{-au}\,du \\
\eta(u,i) :=& \rho\beta\sigma_S - \sigma(u,i) \\
\gamma(u,t,i) :=& \sigma(u,t) - \sigma(u,i).
\end{aligned}
$$

Furthermore, in the Vasicěk model, the price of a zero-coupon bond is an output of the model. The standard expression is

$$
\begin{aligned}
D(t,i) =& \exp\{-A(t,i)r_t + K(t,i)\} \\
A(t,i) =& \frac{1}{a}(1-e^{-a(i-t)}) \\
K(t,i) =& \left(\frac{b}{a} - \frac{\sigma_r^2}{2a^2}\right)(A(t,i)-(i-t)) - \frac{\sigma_r^2(A(t,i))^2}{4a}
\end{aligned}
$$

The second term $\mathcal{Q}(i-1,i)$ is expressed as

$$
\begin{aligned}
\mathcal{Q}(i-1,i) =& \exp\left\{\frac{1}{2}\int\limits_{i-1}^{i} \left(\alpha(\eta(u,i)-\gamma(u,i-1,i)) + (1-\alpha)\sigma_r\mathcal{D}(u,i)\right)^2 du\right\} \\
&+ \exp\left\{\int\limits_{i-1}^{i} \frac{1}{2}\alpha^2\sigma_S^2\beta^2(1-\rho^2)du\right\}
\end{aligned}
$$

and finally the last term $\mathcal{Y}(t,i-1)$ is expressed as

$$
\begin{aligned}
\mathcal{Y}(t,i-1) =& \left(\frac{D(t,i-1)}{D(t,i)}\right)^{\alpha} \exp\Bigg\{(1-\alpha)\mathcal{D}(i-1,i) \\
&\left(e^{-a(i-1-t)}r_t + b\int_t^{i-1} e^{-a(i-1-u)}\,du + \sigma_r\int_t^{i-1} e^{-a(i-1-u)}\,\sigma(u,i)du\right) \\
&- \frac{\alpha}{2}\int_t^{i-1} (\gamma(u,i-1,i))^2\,du\Bigg\} \\
&\cdot \exp\left\{\frac{1}{2}\int_t^{i-1} \left((1-\alpha)\mathcal{D}(i-1,i)\sigma_r e^{-a(i-1-u)} + \alpha\gamma(u,i-1,i)\right)^2 du\right\}
\end{aligned}
$$

PROOF: The proof of the proposition can be found in Appendix 7.1.

The time-t market consistent value of the outstanding liability in benefit scheme II preserves some properties as in scheme I. For instance, in both schemes the market value of the liabilities does not depend on the initial value of the pension fund's assets X_0. Furthermore, for $\alpha = 0$, the time-t value of a periodic benefit payment can now be reduced to

$$\bar{Z}_{t,i}^{II} = Z \cdot E[e^{-\int_t^i r_u du} e^{\int_{i-1}^i r_u du} | \mathcal{F}_t] = Z \cdot D(t, i-1).$$

The same value is obtained for $\beta = 0$. However, no simple analytical conclusions can be drawn concerning the effect of the hybridity indicator α and of the equity ratio β.

In the following section, numerical analyses are conducted to look at the market value of the liabilities in both schemes for different time and examine the effects of the key parameters on this market value.

5. Numerical analysis

In this section we conduct some numerical analysis based on the theoretical valuation model that we have obtained in the previous section. In the numerical examples, we will use the parameters presented in Table I. The parameters concerning the Vasicĕk model are obtained from Brennan and Xia (2002). Note that for $a = 0.63$ and $b = 0.0315$ the long-term mean of the interest rate is $0.0315/0.63 = 0.05$. The initial interest rate level is considered equal to the long-term mean, in order to neutralize the effect of a systemic increase or decrease in interest rates. Below, we illustrate some comparative statistics to examine the effect of the key parameters on the outstanding liability. For both benefit schemes, we examine how the key parameters, specifically the "hybridity" parameter α and the equity allocation β influence the market value of the outstanding liabilities.

5.1. **Benefit scheme I (cumulative returns).** As a first analysis we plot in Figure 1 the time-t value of the total outstanding liability $E[L_t^T|\mathcal{F}_t]$ for $t \in \{1, \cdots, 55\}$ and different values of the hybridity parameter α. The parameter α defines the character of the pension scheme, ranging from a defined benefit scheme for $\alpha = 0$ to a defined contribution for

Variable	Symbol	Value
Number of periods	T	55
Initial liability	Z	1
Flat contribution rate	P_i^x	1
Current interest rate	r_0	0.05
Interest rate volatility	σ_r	0.026
Speed factor	a	0.63
Other parameter in the Vasicěk model	b	0.0315
Long term interest rate	b/a	0.05
Equity volatility	σ_S	0.25
Correlation equity return and interest rate	ρ	-0.129
Allocation to equities	β	0.6
Hybridity parameter	α	0.5

TABLE I. Parameter choices.

$\alpha = 1$. Figure 1 reveals several observations. *First*, the liability value generally decreases over time. This is obvious given the runoff scenario in which the pension population is naturally reduced with the passage of time. *Second*, over time the differences in value due to α disappear. As the time to maturity shortens with the passage of time, the benefits become less uncertain as the period over which the volatility in asset returns can cause significant fluctuations in benefits shortens. The reflection of this is that the differences in value are initially substantial. *Third*, as shown analytically in the previous section, the effect of α is symmetric and non-monotone. For any given t, this leads to a U-shaped function as demonstrated in Figure 2. Both $\alpha = 0.25$ and $\alpha = 0.75$ lead to the same market value, which is also the case for $\alpha = 0$ and $\alpha = 1$. The market value of the outstanding liability decreases in α for $\alpha \in [0, 0.5]$ and increases in α for $\alpha \in [0.5, 1]$. At $\alpha = 0.5$, the lowest value of the outstanding liability results, whereas for $\alpha = 0$ or 1 the maximum value of the outstanding liability follows.

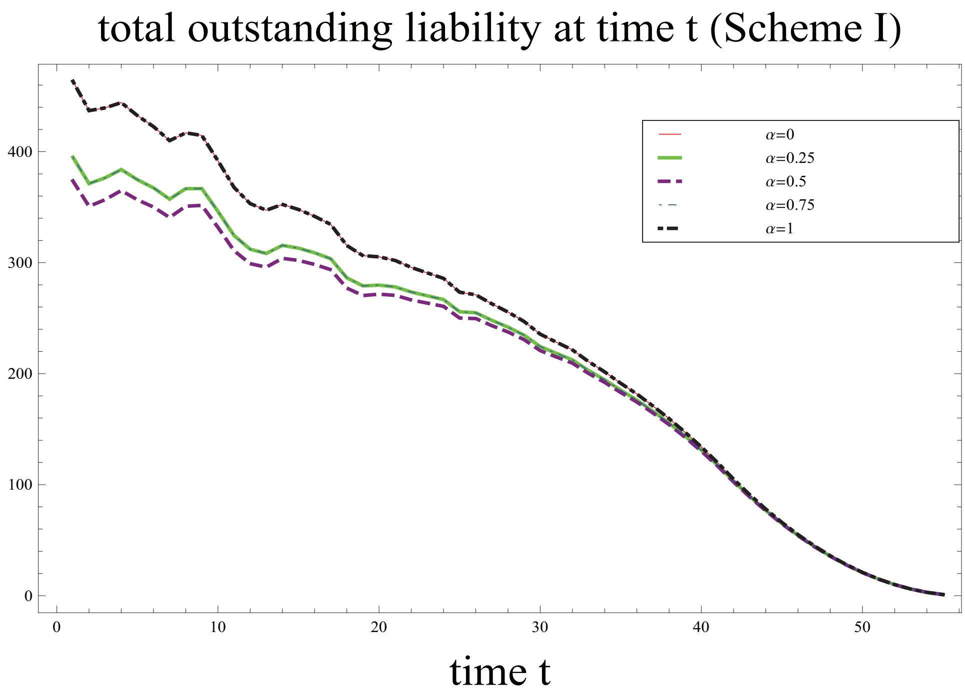

FIGURE 1. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ for different values of the "hybridity" parameter $\alpha = 0$, 0.25, 0.5, 0.75 and 1, using the numerical values from Table I.

As a second sensitivity analysis we explore the impact of the pension fund's asset allocation. Figure 4 shows how the market value of the outstanding liability $E[L_t^T|\mathcal{F}_t]$ depends on the equity allocation β in the pension fund's investment strategy. The riskiness of the pension fund's assets increases by β. As a result, the hybrid pension benefit is more likely to be lower for a fund with a high β than for a pension fund with a low β. Hence, there is a negative relationship between the liability value and the equity allocation β, as shown in Figure 3. This coincides with the rationale derived in the theoretical section of this paper. This however does not imply that employers can get away cheap by increasing the risk profile of the pension fund assets. In an efficient labor market employees will negotiate a compensation for the loss in value until the total labor compensation equals labor productivity. This feedback loop is not presented here.

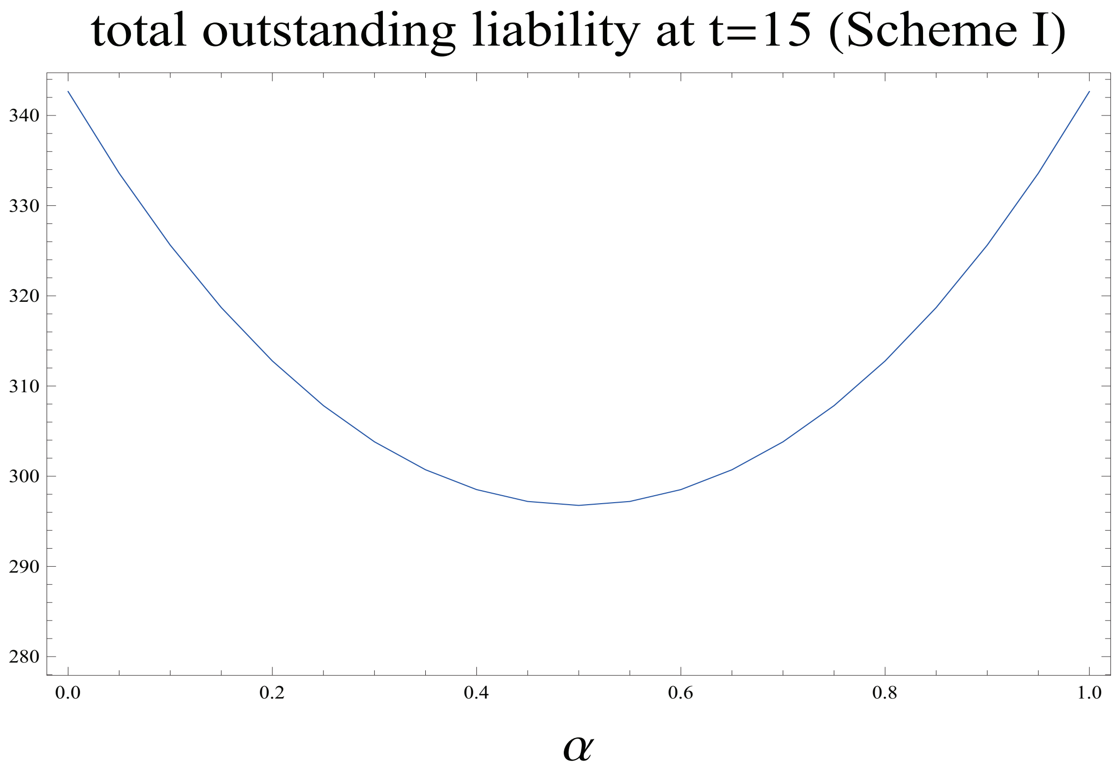

FIGURE 2. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ as a function of α for $t = 15$ using the numerical values from Table I.

Finally, we point out that the effect of asset volatility σ_S on the value of the liabilities is similar to that of the equity allocation β. There is a negative relation between σ_S and the market value of the outstanding liability. This is a standard result in finance: less certain cash flows have a lower value. We do not report the numerical results of different asset volatilities here.

5.2. **Benefit scheme II (period returns).** We now discuss the second benefit scheme, where the benefits are adjusted according to periodic returns. Figure 5 reveals the time-t value of the total outstanding liability $E[L_t^T|\mathcal{F}_t]$ for $t \in \{1, \cdots, 55\}$ and different levels of the hybridity parameter α.

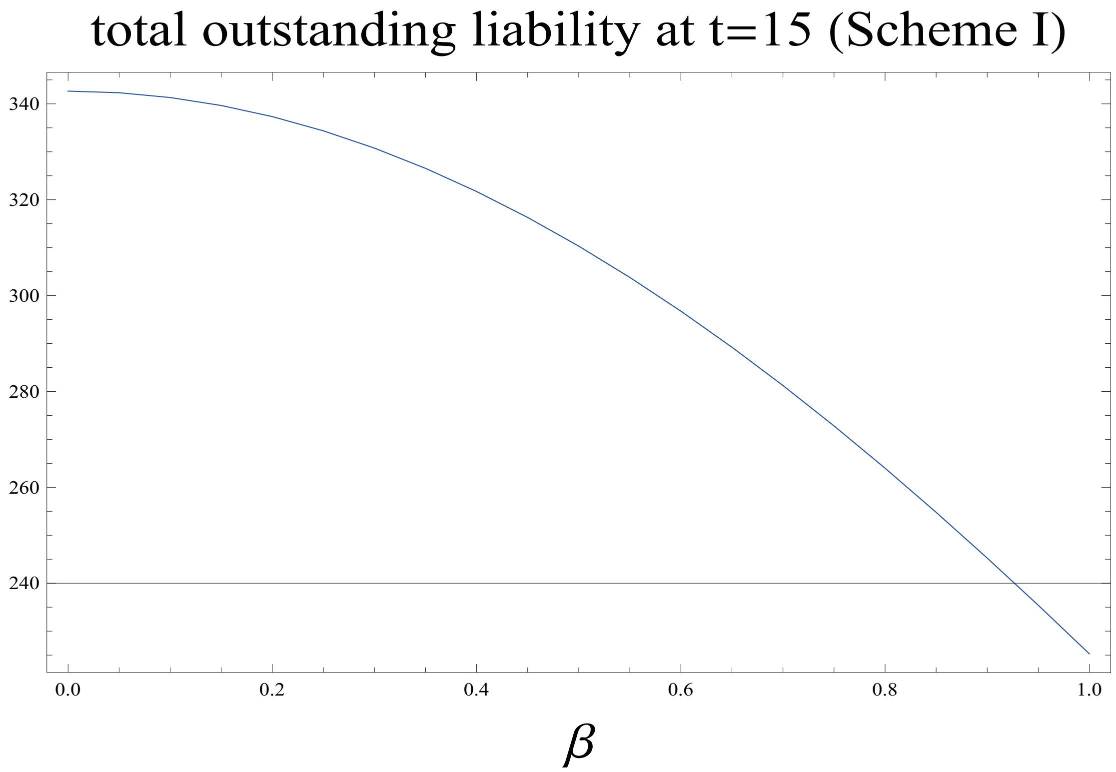

FIGURE 3. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ as a function of β for $t = 15$ using the numerical values from Table I.

First, the market value of the outstanding liability demonstrates quite similar curves to those seen in benefit scheme I. *Second*, the effect of the hybridity parameter α is non-symmetric and does not lead to a U-shaped curve. In scheme II, the effect of α does seem to be monotonic. An increase in α means that the benefit adjustments become more risky, hence their value reduces as is shown in Figure 6.

We have also examined the impact of the pension fund's asset allocation on the value of the liabilities in scheme II. Figure 8 plots how the market value of the outstanding liability depends on the equity allocation β in the pension fund's investment strategy. The higher the β, the more risky the pension fund's assets will be and, consequently, the lower the value of the liabilities. This can also be observed in Figure 7. What is striking is that the effect of the investment strategy under the periodic adjustment mechanism does not seem to have a material impact.

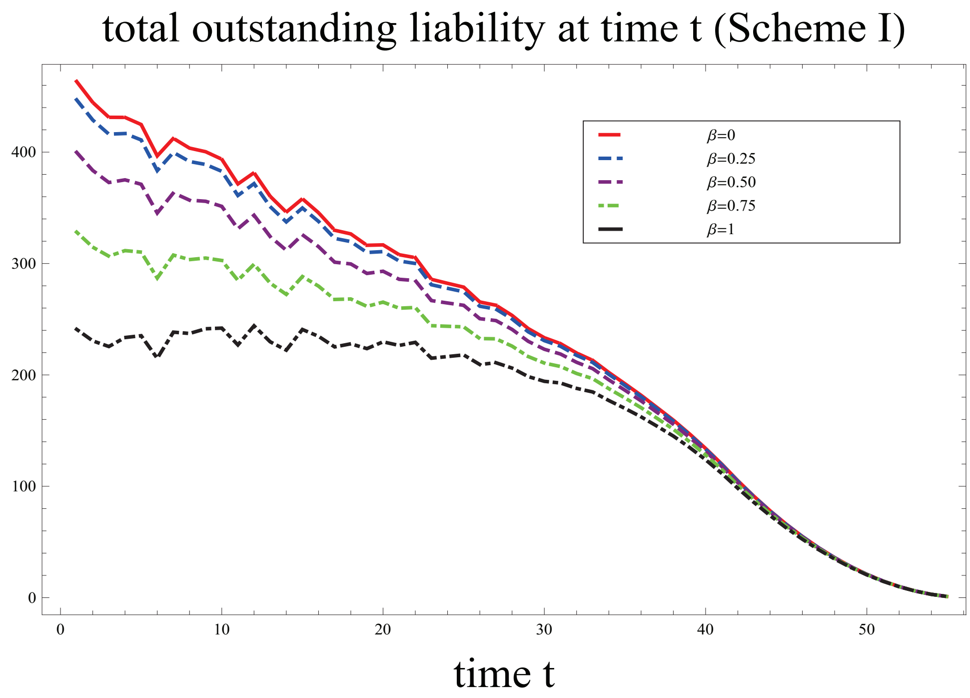

FIGURE 4. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ for different allocations to risky assets β =0, 0.25, 0.5, 0.75 and 1. The "hybridity" parameter is set at $\alpha = 0.5$, using the numerical values from Table I.

6. Conclusion

Market consistent valuation of pension liabilities has received considerable attention in financial economics recently. It is of crucial importance to stakeholders in optimal decision making. Valuation of defined benefit pension liabilities is straightforward and they can be replicated using cash flows from government bonds and other low-default fixed income securities, such as swaps. For pure DC schemes, the value of the liabilities equals the value of the assets. In this paper we explore a model for the market consistent value of pension liabilities in hybrid pension schemes. Thereto we define the return on hybrid liabilities as a linear combination of the return on the pension fund's assets and the risk-free return.

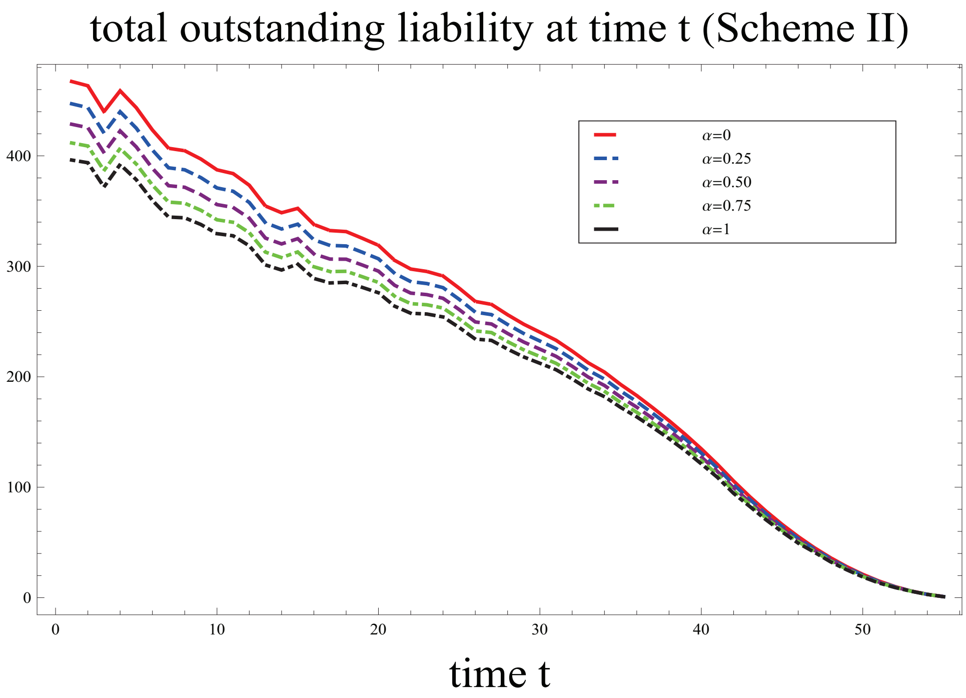

FIGURE 5. Outstanding laibility $E[L_t^T|\mathcal{F}_t]$ for different values of the "hybridity" parameter $\alpha = 0$, 0.25, 0.5, 0.75 and 1, using the numerical values from Table I.

We distinguish between two different pension schemes. In the first scheme, the pension benefit depends on the cumulative return over the entire period from today to the time of payment. In the second scheme, the benefits are adjusted only by the periodic returns. The valuation of the first scheme, based on cumulative returns, proves to be straightforward. The valuation of the second scheme is more complex. We show that the appropriate closed-form valuation formula can be broken down into three parts using the forward-risk-adjusted measure.

The theoretical model and the numerical result reveal that the valuation of hybrid pension liabilities is independent of the expected return on equities. However, the valuation of liabilities is influenced by the nature of the contract ("hybridity"), the pension fund's investment strategy and the volatility of the return on the pension fund's assets. In the

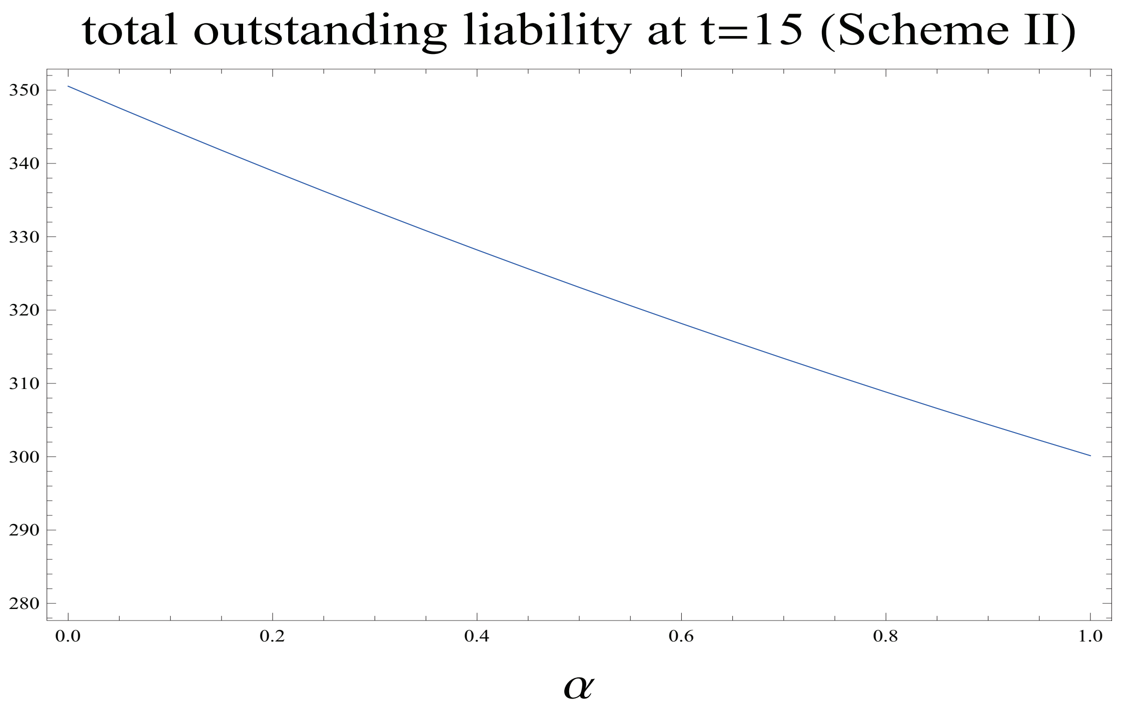

FIGURE 6. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ as a function of α for $t = 15$ using the numerical values from Table I.

first scheme, with cumulative return dependency, the impact of the "hybridity" parameter and the investment policy is much larger than in the second scheme. This should affect the negotiation between employers and employees on total labor compensation.

There are several extensions of the model that are suitable for future research. The scope of the benefit adjustment mechanism may, for instance, be widened. The focus of this paper is on a linear combination of the risk-free return and the portfolio return. Other possible adjustment mechanisms include smoothed returns based on a moving average of past returns, a capped minimum and maximum return ("a collar") or inflation-linked benefits. These will, however, make the valuation model considerably more complex.

7. Appendix

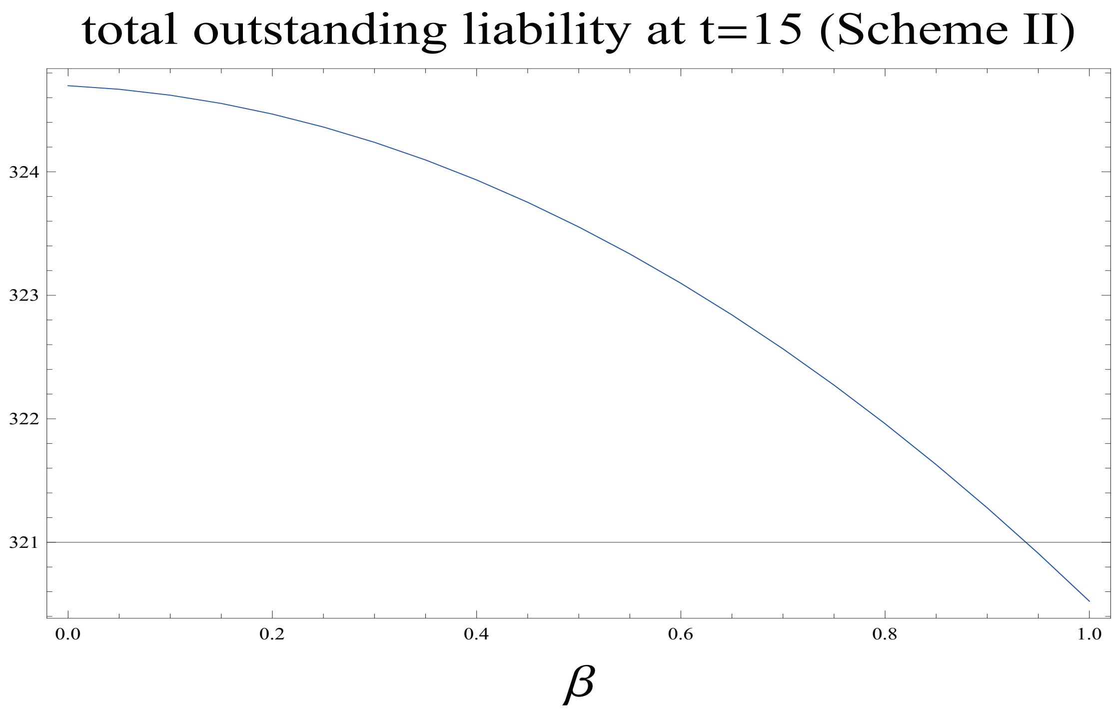

FIGURE 7. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ as a function of β for $t = 15$ using the numerical values from Table I.

7.1. **Appendix.** The vasicěk model for the term structure of interest rates leads to the following asset process of a (default free) zero-coupon bond with maturity $\bar{t}$:

$$dD(t,\bar{t}) = r_t\, D(t,\bar{t})\, dt + \sigma(t,\bar{t})\, D(t,\bar{t})\, dW_t^1, \quad \text{with} \quad D(\bar{t},\bar{t}) = 1,\ P-\text{a.s.}\ \forall\ \bar{t} \in [0,T]$$

The condition $D(\bar{t},\bar{t}) = 1$ describes the fact that the terminal value of a zero-coupon bond equals its face value, here nominated to 1. The volatility $\sigma(t,\bar{t})$ of the zero-coupon bond is a decreasing time-dependent function. Since $D(\bar{t},\bar{t}) = 1$, we define $\sigma(\bar{t},\bar{t}) = 0$. In the Vasicěk model, the volatility of the zero bond is expressed as

$$\sigma(t,\bar{t}) = \frac{\sigma_r}{a}(1 - e^{-a(\bar{t}-t)}),$$

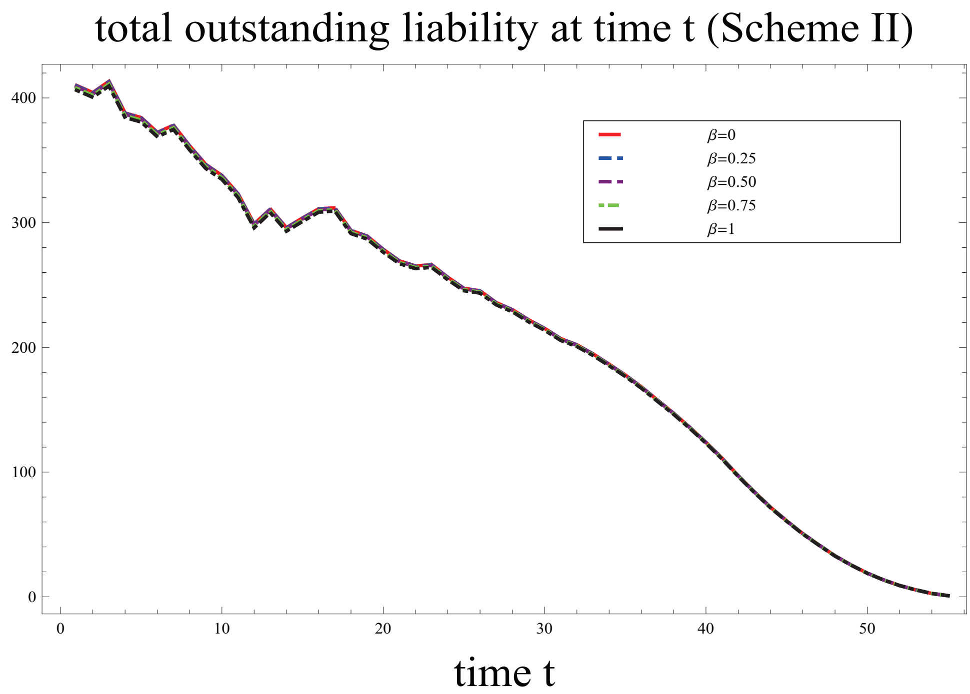

FIGURE 8. Outstanding liability $E[L_t^T|\mathcal{F}_t]$ for different allocations to risky assets β =0, 0.25, 0.5, 0.75 and 1. The "hybridity" parameter is set at $\alpha = 0.5$, using the numerical values from Table I.

Due to the numeraire-invariance principle (Geman *et al.*, 1995), we can express the above expectation under the forward-risk-adjusted measure P^i as follows:

$$\begin{aligned} \bar{Z}_{t,i}^{II} =& Z \cdot E^i \left[\exp\{\alpha(r_i^p - r_{i-1}^p) + (1-\alpha)(R_i - R_{i-1})\} \Big| \mathcal{F}_t \right] \\ \text{with; } \frac{dP^i}{dP}\Big|_{\mathcal{F}_t} =& \exp\left\{ \int_0^t \sigma(u,i) dW_u^1 - \frac{1}{2}\int_0^t \sigma^2(u,i)du \right\}. \end{aligned}$$

After this transformation, the discount factor can be removed from the original expectation. The next step is to write down $\exp\{\alpha(r_i^p - r_{i-1}^p) + (1-\alpha)(R_i - R_{i-1})\} = (X_i/X_{i-1})^\alpha \exp\{(1-\alpha)\int_{i-1}^i r_u du\}$ under the forward-risk-adjusted measure P^i. Since these two terms are correlated, calculating the expectation under P^i is complex. First relying on Ito's Lemma, we can record the forward prices $\frac{X_s}{D(s,i)}$ and $\frac{D(s,\bar{t})}{D(s,i)}$ under P^i

$$d\left(\frac{X_s}{D(s,i)}\right) = \frac{X_s}{D(s,i)}\left((\sigma_S\beta\rho - \sigma(s,i))dW_s^{i1} + \sigma_S\beta\sqrt{1-\rho^2}dW_s^{i2}\right) \tag{6}$$

$$d\left(\frac{D(s,\bar{t})}{D(s,i)}\right) = \frac{D(s,\bar{t})}{D(s,i)}(\sigma(s,\bar{t}) - \sigma(s,i))dW_t^{i1} \tag{7}$$

where $dW_s^{i1} = dW_s^1 - \sigma(s,i)dt$ and $dW_s^{i2} = dW_s^2$ are two independent Brownian motions under the forward-risk-adjusted measure P^i. The solution to (7) for $\bar{t} = s$:

$$\frac{D(s,s)}{D(s,i)} = \frac{D(0,s)}{D(0,i)}\exp\left\{\int_0^s (\sigma(u,s) - \sigma(u,i))\, dW_u^{i1} - \frac{1}{2}\int_0^s (\sigma(u,s) - \sigma(u,i))^2\, du\right\}.$$

Since $D(s,s) = 1$, we can obtain the value of $D(s,i)$. From (6), we have

$$\begin{aligned}\frac{X_s}{D(s,i)} =& \frac{X_0}{D(0,i)}\exp\Big\{\int_0^s (\sigma_S\beta\rho - \sigma(u,i))dW_u^{i1} + \sigma_S\beta\sqrt{1-\rho^2}dW_u^{i2} \\ & - \frac{1}{2}\int_0^s ((\sigma_S\beta\rho - \sigma(u,i))^2 + \sigma_S^2\beta^2(1-\rho^2))du\Big\}.\end{aligned} \tag{8}$$

Subsequently, we can substitute $D(s,i)$ in the solution of (8). As a result we obtain under P^i:

$$\begin{aligned}X_s =& \frac{X_0}{D(0,s)}\exp\left\{-\frac{1}{2}\int_0^s (\eta^2(u,i) + \sigma_S^2\beta^2(1-\rho^2)) - \gamma^2(u,s,i))du\right. \\ & \left. + \int_0^s (\eta(u,i) - \gamma(u,s,i))dW_u^{i1} + \int_0^s \sigma_S\beta\sqrt{1-\rho^2}dW_u^{i2}\right\}\end{aligned} \tag{9}$$

with $\eta(u,i) := \rho\beta\sigma_S - \sigma(u,i), \quad \gamma(u,s,i) := \sigma(u,s) - \sigma(u,i).$

Note that upon the information structure $\mathcal{F}_t$, we can write X_s, $s > t$ as follows:

$$\begin{aligned}X_s =& \frac{X_t}{D(t,s)}\exp\left\{-\frac{1}{2}\int_t^s (\eta^2(u,i) + \sigma_S^2\beta^2(1-\rho^2)) - \gamma^2(u,s,i))du\right. \\ & \left. + \int_t^s (\eta(u,i) - \gamma(u,s,i))dW_u^{i1} + \int_t^s \sigma_S\beta\sqrt{1-\rho^2}dW_u^{i2}\right\}.\end{aligned}$$

Furthermore, in the Vasicěk model, we have

$$\begin{aligned} r_s =& e^{-a(s-t)}r_t + b\int_t^s e^{-a(s-u)}\,du + \sigma_r\int_t^s e^{-a(s-u)}\,dW_u^1 \ \ \text{(under } P) \\ =& e^{-a(s-t)}r_t + b\int_t^s e^{-a(s-u)}\,du + \sigma_r\int_t^s e^{-a(s-u)}\,(dW_u^{i1} + \sigma(u,i)du) \ \ \text{(under } P^i) \end{aligned} \tag{10}$$

$$\begin{aligned} \int_t^s r_u du =& \mathcal{D}(t,s)r_t + b\int_t^s \mathcal{D}(u,s)\,du + \sigma_r\int_t^s \mathcal{D}(u,s)dW_u^1 \ \ \text{(under } P) \\ =& \mathcal{D}(t,s)r_t + b\int_t^s \mathcal{D}(u,s)\,du + \sigma_r\int_t^s \mathcal{D}(u,s)(dW_u^{i1} + \sigma(u,i)du) \ \ \text{(under } P^i) \end{aligned} \tag{11}$$

where $\mathcal{D}(t,s) := e^{at}\int_t^s e^{-au}\,du.$

The market value of the periodic benefit is then expressed as:

$$\begin{aligned} \bar{Z}_{t,i}^{II} =& Z\cdot E^i\left[\exp\{\alpha(r_i^p - r_{i-1}^p) + (1-\alpha)\int_{i-1}^i r_u du\}\Big|\mathcal{F}_t\right] \\ =& Z\cdot E^i\left[E^i\left[\exp\{\alpha(r_i^p - r_{i-1}^p) + (1-\alpha)\int_{i-1}^i r_u du\}\Big|\mathcal{F}_{i-1}\right]\Big|\mathcal{F}_t\right] \\ =& Z\cdot E^i\Bigg[\left(\frac{D(i-1,i-1)}{D(i-1,i)}\right)^\alpha E^i\Big[\exp\Big\{-\frac{\alpha}{2}\int_{i-1}^i ((\eta^2(u,i) + \sigma_S^2\beta^2(1-\rho^2)) - \gamma^2(u,i-1,i))du \\ &+ \alpha\int_{i-1}^i (\eta(u,i) - \gamma(u,i-1,i))dW_u^{i1} + \alpha\int_{i-1}^i \sigma_S\beta\sqrt{1-\rho^2}dW_u^{i2}\Big\} \\ &\exp\left\{(1-\alpha)\left(\mathcal{D}(i-1,i)r_{i-1} + b\int_{i-1}^i \mathcal{D}(u,i)\,du + \sigma_r\int_{i-1}^i \mathcal{D}(u,i)(dW_u^{i1} + \sigma(u,i)du)\right)\right\}|\mathcal{F}_{i-1}\Big]\Big|\mathcal{F}_t\Bigg]. \end{aligned}$$

The above expectation can be rewritten as

$$\bar{Z}_{t,i}^{II} = E^i\left[\mathcal{M}(i-1,i)\mathcal{Q}(i-1,i)\frac{1}{(D(i-1,i))^\alpha}\exp\{(1-\alpha)\mathcal{D}(i-1,i)r_{i-1}\}\Big|\mathcal{F}_t\right]$$

where we have

$$
\begin{aligned}
\mathcal{M}(i-1,i) =& Z\cdot\exp\Big\{-\frac{\alpha}{2}\int\limits_{i-1}^{i}((\eta^2(u,i)+\sigma_S^2\beta^2(1-\rho^2))-\gamma^2(u,i-1,i))du\Big\}\\
&\cdot\exp\left\{(1-\alpha)\left(b\int_{i-1}^{i}\mathcal{D}(u,i)\,du+\sigma_r\int_{i-1}^{i}\mathcal{D}(u,i)\sigma(u,i)du\right)\right\}\\
\mathcal{Q}(i-1,i) =& \exp\left\{\frac{1}{2}\int\limits_{i-1}^{i}\left(\alpha(\eta(u,i)-\gamma(u,i-1,i))+(1-\alpha)\sigma_r\mathcal{D}(u,i)\right)^2du\right\}\\
&+\exp\left\{\int\limits_{i-1}^{i}\frac{1}{2}\alpha^2\sigma_S^2\beta^2(1-\rho^2)du\right\}
\end{aligned}
$$

Since $\mathcal{M}(i-1,i)$ and $\mathcal{Q}(i-1,i)$ are deterministic, this results in

$$
\begin{aligned}
\bar{Z}^{II}_{t,i} =& \mathcal{M}(i-1,i)\mathcal{Q}(i-1,i)\cdot E^i\left[\frac{1}{(D(i-1,i))^\alpha}\exp\left\{(1-\alpha)\mathcal{D}(i-1,i)r_{i-1}\right\}\Big|\mathcal{F}_t\right]\\
=& \mathcal{M}(i-1,i)\mathcal{Q}(i-1,i)\cdot E^i\Big[\left(\frac{D(t,i-1)}{D(t,i)}\right)^\alpha\exp\Big\{\int_t^{i-1}\alpha\gamma(u,i-1,i)dW_u^{i1}\\
&-\frac{1}{2}\alpha\int_t^{i-1}(\gamma(u,i-1,i))^2du\Big\}\exp\Big\{(1-\alpha)\mathcal{D}(i-1,i)\\
&\left(e^{-a(i-1-t)}r_t+b\int_t^{i-1}e^{-a(i-1-u)}du+\sigma_r\int_t^{i-1}e^{-a(i-1-u)}\left(dW_u^{i1}+\sigma(u,i)du\right)\right)\Big\}\Big|\mathcal{F}_t\Big]\\
=& \mathcal{M}(i-1,i)\mathcal{Q}(i-1,i)\cdot\left(\frac{D(t,i-1)}{D(t,i)}\right)^\alpha\exp\Bigg\{(1-\alpha)\mathcal{D}(i-1,i)\\
&\left(e^{-a(i-1-t)}r_t+b\int_t^{i-1}e^{-a(i-1-u)}du+\sigma_r\int_t^{i-1}e^{-a(i-1-u)}\sigma(u,i)du\right)-\frac{\alpha}{2}\int_t^{i-1}(\gamma(u,i-1,i))^2du\Bigg\}\\
&\exp\left\{\frac{1}{2}\int_t^{i-1}\left((1-\alpha)\mathcal{D}(i-1,i)\sigma_r e^{-a(i-1-u)}+\alpha\gamma(u,i-1,i)\right)^2du\right\}\\
:=& \mathcal{M}(i-1,i)\mathcal{Q}(i-1,i)\mathcal{Y}(t,i-1).
\end{aligned}
$$

Furthermore, in the Vasicěk model, the price of a zero-coupon bond $D(t,i)$ is an output of the model and is expressed in the usual way as

$$
\begin{aligned}
D(t,i) &= \exp\{-A(t,i)r_t + K(t,i)\} \\
A(t,i) &= \frac{1}{a}(1 - e^{-a(i-t)}) \\
K(t,i) &= \left(\frac{b}{a} - \frac{\sigma_r^2}{2a^2}\right)(A(t,i) - (i-t)) - \frac{\sigma_r^2 (A(t,i))^2}{4a}
\end{aligned}
$$

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