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\* Views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.

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# Why risk is so hard to measure\*

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## Abstract

This paper analyses the accuracy and reliability of standard techniques for risk analysis used by the financial industry as well as in regulations. We focus on the difference between value-at-risk and expected shortfall, the small sample properties of these risk measures and the impact of using an overlapping approach to construct data for longer holding periods. Overall, we find that risk forecasts are extremely uncertain at low sample sizes. By comparing the estimation uncertainty, we find that value-at-risk is superior to expected shortfall and the time-scaling approach for risk forecasts with longer holding periods is preferable to using overlapping data.

**Keywords:** Value-at-risk, expected shortfall, finite sample properties, Basel III.

**JEL classifications:** C10, C15, G18.

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# 1 Introduction

Financial risk is usually forecasted with sophisticated statistical methods. However, in spite of their prevalence in industry applications and financial regulations, the performance of such methods is poorly understood. This is a concern since minor variations in model assumptions can lead to vastly different risk forecasts for the same portfolio, forecasts that are all equally plausible *ex-ante*. These results have problematic implications for many practical applications, especially where the cost of type I and type II error is not trivial.

Our aim in this paper is to analyze the most common practices in market risk modeling, identify under what conditions they deliver reliable answers, when and how they fail to live up to expectations and ultimately make recommendations to practitioners and regulators on their proper use. In particular, we focus on the three main challenges arise in the forecasting of financial risk: the choice of risk measure, data sample and statistical method.

A large number of statistical methods for forecasting risk have been proposed, but as a practical matter, only a handful have found significant traction, as discussed in Danielsson et al. (2015). Of these, all but one depend on some parametric model, while one, historical simulation (HS), is model independent. Our objective in this paper is not to compare and contrast the various risk forecast methods: after all, a large number of high-quality papers exist on this very topic. Instead, we want to see how a representative risk forecast method performs, identifying results that are related to the

technical choice on the other two key issues: risk measure and data.

Given our objectives, it is appropriate to focus on HS, not only is it a commonly used method, for example, 60% of the US banks considered O'Brien and Szerszen (2014) use HS. More fundamentally, the good performance of a specific parametric model is usually driven by the fact that the model is close to the data generating process (DGP) and it is not possible to find a parametric model that performs consistently well across all DGPs. Although HS is the simplest estimation method, it has the advantage of not being dependent on a particular parametric DGP, and any other method would be biased towards its DGP, creating an unlevel playing field.

While our first contribution is the practical comparison of Value-at-Risk (VaR) to expected shortfall (ES).<sup>1</sup> A common view holds that VaR is inherently inferior to ES, a view supported by three convincing arguments. First, VaR is not a coherent measure unlike ES, as noted by Artzner et al. (1999). Second, as a quantile, VaR is unable to capture the risk in the tails beyond the specific probability, while ES accounts for all tail events. Finally, it is easier for financial institutions to manipulate VaR than ES. Perhaps swayed by the theoretical advantages, ES appears increasingly preferred both by practitioners and regulators, most significantly expressed in Basel III. While the Proposal is light on motivation, the little that is stated only refers to theoretic tail advantages. The practical properties of both ES and VaR, such as

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<sup>1</sup>Ever since its introduction by RiskMetrics Group (1993) and especially the incorporation into financial regulations by the Basel Committee (1996), VaR is the most commonly used market risk measure. While it has come under considerable criticism, it has generally been preferred to other measures such as Artzner's et al. (1999) ES, partly because of the work of Yamai and Yoshida (2002, 2005).

their estimation uncertainties, are less understood, especially since empirically there is no reason to believe that the preference ordering is as clear-cut. After all, implementation introduces additional considerations, some that work in opposite directions. The estimation of ES requires more steps and more assumptions than the estimation of VaR, giving rise to more estimation uncertainty. However, ES smooths out the tails and therefore might perform better in practice.

Our second contribution is to investigate how best to use data. Many applications require the estimation of risk over multiple day holding horizons. There are three ways one can calculate such risk, use non-overlapping data, use overlapping data, or timescale daily risk forecasts. While the first is generally preferred, it may not be possible because the lack of data.

In our third and final contribution we study whether the estimation of risk measures is robust when considering small — and typical in practical use — sample sizes. Although the asymptotic properties of risk measures can be established using statistical theories, known asymptotic properties of the risk forecast estimators might be very different in typical sample sizes. Estimation uncertainty based on asymptotic theory can therefore be misleading for small sample size analysis.

We address each of these three questions from both theoretic and empirical points of view. Ideally, one would evaluate the robustness of risk analysis with real data, but that is challenging because we do not know the true DGP of the observed data and neither do we have any assurance that data across time and assets maintains consistent statistical properties. We therefore start

with theoretic and simulation analysis, which allows us to study properties of the risk measures when we know the DGP. We then augment the analysis by results from the Center for Research in Security Prices (CRSP) universe of stock returns.

While the theoretic analysis directly relates to the vast extant literature on risk measures, we found practically no work on the finite sample analysis of the practical properties of risk measures exist. We surmise that an important reason relates to computational difficulties, especially the very large simulation size needed. We are estimating not only the risk measures but also the uncertainty of those estimates, where for example, we need to capture the “quantiles of the quantiles”. To achieve robust results, in the sense that they are accurate up to three significant digits, one needs to do ten million Monte Carlo replications, each of various sizes.

We obtain four sets of results.

First, VaR and ES are in most cases related by a small constant implying that they are equally informative. In the special case of Basel III, the 97.5% ES is essentially the same as the 99% VaR. This suggests that even if ES is theoretically better at capturing the tails, in practice one might just multiply VaR by a small constant to get ES.

Second, ES is estimated with more uncertainty than VaR, both when estimated each at the same probability levels and also when using the Basel III combination, ES(97.5%) and VaR(99%).

Third, as the sample size gets smaller, the estimation uncertainty of both

VaR and ES becomes extremely large. At the smallest samples, often the most commonly used in practice, the uncertainty is so large that the risk forecast is essentially indistinguishable from random noise. It takes a sample of half a century of daily observations for the numerical estimators to achieve their asymptotic properties. The confidence interval around the risk forecasts is very far from being symmetric, the upper 99% confidence bound is a multiple of the forecast, which obviously cannot be the case for the lower confidence bound. This means that if one uses the standard error as a measure of uncertainty, it will be strongly biased downwards.

In our final result, we compare the square-root-of-time approach to the overlapping approach and find that the overlapping approach results in more estimation uncertainty.

Taken together, these results suggest that common practices and trends in risk management are misguided. Unless there is a specific reason to do so, VaR is preferred to ES, and the overlapping approach should be avoided. Most importantly, risk forecasts at commonly used sample sizes are virtually indistinguishable from random noise. To us, it is a concern that vast amounts of resources are allocated based on such flimsy evidence. This is especially problematic in the calculation of banks' capital requirement for market risk. A financial institution has the flexibility internally to tailor the risk analysis exploiting the problems outlined here.

The structure of the paper is as follows. We discuss the theoretic properties of the risk measures in Section 2 and the Monte Carlo results in Section 3. Empirical results using stock returns in the CRSP database are presented

in Section 4. The conclusion and implications of our analysis are discussed in Section 5. Mathematical proofs are relegated to the Appendix.

## 2 Estimation uncertainty

Many authors have considered the various theoretical properties of risk measures and corresponding estimation methods. Here we are interested in three aspects of risk analysis that see little coverage in the extant literature: The relationship between ES and VaR, the impact of using the overlapping approach and the small sample properties of the risk measures. We consider both the case where the probability for VaR and ES is the same, and also the Basel III case where the comparison is between ES(97.5%) and VaR(99%).

Denote the profit and loss of a trading portfolio as  $PL$  and let  $X \equiv -PL$ , so we can indicate a loss by a positive number. Suppose that  $X$  follows a distribution function  $F$  and we obtain a sample of size  $N$  from that distribution. With denoting the probability level by  $p$ , we refer to VaR and ES by  $q_F := q_F(p)$  and  $e_F := e_F(p)$ , respectively. We estimate the two risk measures by HS as follows. Rank the  $N$  observations as  $X_{N,1} \leq X_{N,2} \leq \dots \leq X_{N,N}$ . Then

$$\begin{aligned} \hat{q}_F &= X_{N,[pN]}, \\ \hat{e}_F &= \frac{1}{(1-p)N} \sum_{j=1}^{(1-p)N} X_{N,N-j+1}. \end{aligned} \tag{1}$$

Asymptotically, these estimators are unbiased. But a well-known result,

dating at least back to Blom (1958), finds that quantile estimators using HS are biased with finite sample sizes: the probability of exceeding the quantile  $X_{N,[pN]}$  is slightly lower than  $p$ . It is straightforward to adjust for this bias, using for example the methods proposed by Hyndman and Fan (1996).

## 2.1 VaR and ES

### 2.1.1 The levels of VaR and ES

Consider the ratio of ES to VaR,  $e_F/q_F$ , for a range of distribution functions  $F$ . Starting with the Gaussian, with mean zero and standard deviation  $\sigma$ , then  $q_F = \sigma q_{N(0,1)}$  and  $e_F = \sigma e_{N(0,1)}$ , where  $N(0,1)$  denotes the standard normal distribution. It is obvious that for the same  $p$  levels,  $e_F^{(p)}/q_F^{(p)} > 1$  and it is straightforward to verify that:

$$\lim_{p \rightarrow 1} \frac{e_F(p)}{q_F(p)} = 1.$$

In other words, as probabilities become more extreme, the relative difference between ES and VaR diminishes. At a finite level, such a ratio can be explicitly calculated, for example,  $e_F^{(0.99)}/q_F^{(0.99)} = 1.146$ . When considering different  $p$  levels for the two risk measures, such as in comparing the Basel III proposal, we get  $e_F^{(0.975)}/q_F^{(0.99)} = 1.005$ . Hence the two risk measures are roughly identical under normality.

Since financial returns exhibit heavy tails, a more realistic analysis uses some heavy tailed distribution, such as the Student-t distribution. Similar to the normal case, it is straightforward to calculate the ratio of ES to VaR for

the Student–t distribution with any particular degrees of freedom, and probability levels. For example, consider the Student–t with degrees of freedom three. Then  $e_F(0.99)/q_F(0.99) = 1.54$  and  $e_F(0.975)/q_F(0.99) = 1.11$ .

However, we are more interested in a general expression of the relationship between VaR and ES, one that applies to all heavy–tailed distributions. To this end, we make use of Extreme Value Theory (EVT), in which a heavy–tailed distribution is defined by regular variation. That is,

$$\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha},$$

for some  $\alpha > 0$ , known as the tail index. For the Student–t distribution, the tail index equals the degrees of freedom. Note that the assumption of regular variation only applies to the right tail of  $F$ , and thus does not impose any restriction on the rest of the distribution, allowing this approach to capture a large range of models. Indeed, an assumption of regular variation is sufficient for inference on tail risk measures.

The following proposition gives the theoretical foundation for comparing the levels of VaR and ES at high probability levels. The proof is postponed to the Appendix.

**Proposition 1** *Suppose  $F$  is a heavy–tailed distribution with tail index  $\alpha$ . Given any constant  $c > 0$ , we have that*

$$\lim_{s \rightarrow 0} \frac{e_F(1 - cs)}{q_F(1 - s)} = \frac{\alpha}{\alpha - 1} c^{-1/\alpha}.$$

To compare the VaR and ES with the same probability level, one can take  $c = 1$  and  $s = 1 - p$  in Proposition 1, and get that

$$\lim_{p \rightarrow 1} \frac{e_F(p)}{q_F(p)} = \frac{\alpha}{\alpha - 1}.$$

That is, when the probability is the same for both risk measures, the ES is equivalent to the VaR times the multiplier  $\alpha/(\alpha-1)$ . This ratio is higher than one, which gives the essential difference between heavy-tailed distributions and thin-tailed distributions such as the normal distribution. Since the multiplier is decreasing in  $\alpha$ , the more heavy-tailed the distribution of  $F$ , the larger the difference between ES and VaR.

To compare the VaR(99%) with ES(97.5%), set  $c$  and  $s$  in Proposition 1 such that  $s = 1\%$  and  $cs = 2.5\%$ . Then;

$$\frac{e_F(p_2)}{q_F(p_1)} \approx \frac{\alpha}{\alpha - 1} (2.5)^{-1/\alpha} := f(\alpha).$$

That is, when comparing ES(97.5%) to VaR(99%), the ratio is given by the function  $f(\alpha)$ . We plot this function in Figure 1 for  $\alpha$  ranging from 2 to 5, which is more than wide enough to cover tail thicknesses commonly observed. Note the ratio is decreasing in  $\alpha$ , ranging between 1.105 and 1.041 for  $\alpha$  ranging from 3 to 5.

### 2.1.2 The estimation uncertainty of VaR and ES

In what follows, we focus our attention on the best case scenario where the data is i.i.d. and we know it is i.i.d. If we also had to estimate the dynamic

structure, the estimation uncertainty would be further increased. We focus our attention on the case where  $F$  is heavy-tailed with a tail index  $\alpha$ , with the Gaussian as the special case where  $\alpha = +\infty$ .

We only consider the HS method, and derive the asymptotic properties of two estimators,  $\hat{q}_F$  and  $\hat{e}_F$ , as given in (1). In HS estimation, only the top  $(1-p)N$  order statistics are used.

Denote the number of observations used in estimators (1) as  $k_q := k_q(N)$  and  $k_e := k_e(N)$ , such that  $k_q, k_e \rightarrow \infty$ ,  $k_q/N \rightarrow 0$ ,  $k_e/N \rightarrow 0$  as  $N \rightarrow \infty$ . We can then generalize (1) by defining the ES and VaR as:

$$\begin{aligned}\hat{q}_F(1 - k_q/N) &= X_{N, N-k_q}, \\ \hat{e}_F(1 - k_e/N) &= \frac{1}{k_e} \sum_{j=1}^{k_e} X_{N, N-j+1}.\end{aligned}$$

The following proposition gives the asymptotic properties of these estimators for a general  $k$  sequence. The proof is postponed to the Appendix.

**Proposition 2** *Suppose that  $X_1, \dots, X_N$  are i.i.d. and drawn from a heavy tailed distribution function  $F$  with  $\alpha > 2$ . Denote  $U = (1/(1-F))^\leftarrow$  as the quantile function. Assume the usual second order condition holds:*

$$\lim_{t \rightarrow \infty} \frac{\frac{U(tx)}{U(t)} - x^{1/\alpha}}{A(t)} = x^{1/\alpha} \frac{x^\rho - 1}{\rho},$$

for a constant  $\rho \leq 0$  and a function  $A(t)$  such that  $\lim_{t \rightarrow \infty} A(t) = 0$ . Suppose  $k := k(N)$  is an intermediate sequence such that as  $N \rightarrow \infty$ ,  $k \rightarrow \infty$ ,  $k/N \rightarrow 0$  and  $\sqrt{k}A(N/k) \rightarrow \lambda$  with a constant  $\lambda$ . Then, we have that as

$N \rightarrow \infty$ ,

$$\begin{aligned}\sqrt{k} \left( \frac{\hat{q}_F(1 - k/N)}{q_F(1 - k/N)} - 1 \right) &\xrightarrow{d} N \left( 0, \frac{1}{\alpha^2} \right), \\ \sqrt{k} \left( \frac{\hat{e}_F(1 - k/N)}{e_F(1 - k/N)} - 1 \right) &\xrightarrow{d} N \left( 0, \frac{2(\alpha - 1)}{\alpha^2(\alpha - 2)} \right).\end{aligned}$$

From Proposition 2, both estimators are asymptotically unbiased. We focus on comparing their asymptotic variances.

First, we consider the case in which the ES and VaR probability is the same. Consider,  $k_e = k_q = (1 - p)N$  with  $p \rightarrow 1$  as  $N \rightarrow \infty$ . Consequently, we get that

$$\frac{\text{Var} \left( \frac{\hat{e}_F(p)}{e_F(p)} \right)}{\text{Var} \left( \frac{\hat{q}_F(p)}{q_F(p)} \right)} \approx \frac{\frac{2(\alpha-1)}{\alpha^2(\alpha-2)} \frac{1}{k_e}}{\frac{1}{\alpha^2} \frac{1}{k_q}} = \frac{2(\alpha - 1)}{\alpha - 2} = 1 + \frac{\alpha}{\alpha - 2}.$$

Which means that when considering the same probability level, the relative estimation uncertainty of the ES measure is higher than that of the VaR measure. The difference is larger for lower  $\alpha$ , i.e. heavier distributions.

Next, we compare the estimation uncertainty of VaR(99%) and ES(97.5%). In this case, we need to set  $k_e/k_q = (1 - p_2)/(1 - p_1) = 2.5$ , which reflects the relative difference in the two tail probabilities. By applying the Proposition with  $k_q$  and  $k_e$  such that  $k_e/k_q = 2.5$ , we get that

$$\frac{\text{Var} \left( \frac{\hat{e}_F(p_2)}{e_F(p_2)} \right)}{\text{Var} \left( \frac{\hat{q}_F(p_1)}{q_F(p_1)} \right)} \approx \frac{\frac{2(\alpha-1)}{\alpha^2(\alpha-2)} \frac{1}{k_e}}{\frac{1}{\alpha^2} \frac{1}{k_q}} = \frac{4(\alpha - 1)}{5(\alpha - 2)} =: g(\alpha).$$

The function  $g(\alpha)$  is decreasing in  $\alpha$ . By solving  $g(\alpha) = 1$ , we get the

break-even point at  $\alpha^{\text{be}} = 6$ . For  $\alpha > 6$ ,  $g(\alpha) < 1$ ; if  $\alpha < 6$ , then  $g(\alpha) > 1$ .

That means, if the losses are heavy-tailed with  $\alpha < 6$ , the estimation uncertainty in ES(97.5%) is higher than that of VaR(99%).

## 2.2 Overlapping and time-scaling approaches

Consider the problem of forecasting risk over holding periods longer than one day, say a  $H$  days and suppose data is observed at the daily frequency. There are three ways one can obtain multi-day holding period risk forecasts: First, estimate a daily risk forecast and apply some scaling law to obtain the multi-day forecast, typically square-root-of-time. We call this approach the *time-scaling approach*. Alternatively one can time aggregate daily observations. Focussing on log returns, a 10 day return would be the sum of 10 one-day returns, and the risk forecasts would then be made by these 10 day returns. Here we have two alternatives. We can either use non-overlapping aggregated data, or allow the aggregation periods to overlap. We term the first the *non-overlapping approach* and the second the *overlapping approach*.

Ideally, the non-overlapping approach is preferred, but in practice it would likely result in excessively large data requirements, beyond what would be available in most cases. This means that any implementation needs to depend on either the time-scaling approach or overlapping approach. From a purely theoretic point of view, the time-scaling approach is not very attractive, the common square-root-of-time approach is only correct for scaling VaR or ES for independently and identically distributed (i.i.d.) nor-

mal returns, and in practice is either higher or lower depending on the unknown underlying stochastic process. This suggests that the overlapping approach might be preferred. By aggregating high frequency observations we get smoother forecasts due to the smoothing of what might be seen as anomalous extreme outcomes. In addition, when dealing with infrequent trading, high-frequency (daily) observations become unreliable.

Our purely anecdotal observation of practitioners suggests that using the overlapping approach is increasingly preferred to the scaling method. The Basel Committee on Banking Supervision (BSBC), as expressed in the 2013 version of the Basel III Proposals, suggested the overlapping approach, but not in the revised 2014 version.. However, the overlapping approach gives rise to a particular theoretical challenge, induced dependence in the constructed dataset, and hence the potential to increase the estimation uncertainty. The pros and cons of using the overlapping approach for forecasting risk have until now been mostly conjectured, and not been supported by analytical work. While some theoretical results exist on the properties of square-root-of-time approach compared to overlapping approach, little to none exists on the impact on the estimation uncertainties.

Suppose  $Y_1, Y_2, \dots, Y_{N+H-1}$  are i.i.d. daily observations with the common distribution function  $G$ . We can then define the two alternatives by;

**The overlapping approach** Calculate overlapping observations by

$$Z_i = \sum_{j=1}^H Y_{i+j-1}.$$

Denote the distribution of  $Z_i$  by  $F$ . Then use  $Z_1, \dots, Z_N$  in (1) to estimate  $q_F$  and  $e_F$ .

**The square-root-of-time approach** Use  $Y_1, \dots, Y_N$ , to estimate  $q_G$  and  $e_G$  from (1). Then we estimate  $q_F$  and  $e_F$  by  $\sqrt{H}\hat{q}_G$  and  $\sqrt{H}\hat{e}_G$  respectively.

The number of observations used in these approaches is  $N + H - 1$  and  $N$ , respectively so the required sample sizes are comparable. The overlapping approach provides a direct estimate of  $q_F$ , while the time scaling approach only provides an estimate of  $\sqrt{H}q_G$ , which is an approximation of  $q_F$ . In practice, this approximation turns to be an exact relation if  $F$  is i.i.d. normal, and slightly too high for i.i.d. heavy-tailed distributions.

Consider the overlapping approach. In this case, the  $H$ -day observations  $Z_1, \dots, Z_N$  are not independent but exhibit a moving average. Clearly, if  $G$  is Gaussian, so is  $F$ . If  $G$  is heavy-tailed with tail index  $\alpha$ ,  $F$  will also be heavy-tailed with tail index  $\alpha$ ; see Feller (1971).

The estimation uncertainty of risk measures based on dependent observations is given in the following Proposition. Again, the proof is postponed to the Appendix.

**Proposition 3** *Suppose  $Z_1, \dots, Z_N$  are dependent observations defined as  $Z_i = \sum_{j=1}^H Y_{i+j-1}$ , where  $Y_1, Y_2, \dots, Y_{N+H-1}$  are i.i.d. observations from a heavy tailed distribution function with  $\alpha > 2$ . Suppose the distribution function  $G$  satisfies the second order condition in Proposition 2 and the sequence*

$k$  satisfies the same condition in Proposition 2. We have that as  $N \rightarrow \infty$ ,

$$\begin{aligned} \sqrt{k} \left( \frac{\hat{q}_F(1 - k/N)}{q_F(1 - k/N)} - 1 \right) &\xrightarrow{d} N \left( 0, H \frac{1}{\alpha^2} \right), \\ \sqrt{k} \left( \frac{\hat{e}_F(1 - k/N)}{e_F(1 - k/N)} - 1 \right) &\xrightarrow{d} N \left( 0, H \frac{2(\alpha - 1)}{\alpha^2(\alpha - 2)} \right). \end{aligned}$$

Proposition 3 shows that using overlapping data enlarges the estimation variance for the estimators on both VaR and ES by a factor proportional to  $H$ , leading to the following corollary on the comparison of estimation variance across the strategies.

**Corollary 4** *As  $N \rightarrow \infty$ , for a given  $k$  sequence satisfying the conditions in Proposition 3, we have that*

$$\text{Var} \left( \frac{\hat{q}_F(k/N)}{q_F(k/N)} \right) \sim H \text{Var} \left( \frac{\sqrt{H} \hat{q}_G(k/N)}{\sqrt{H} q_G(k/N)} \right).$$

A similar results holds for ES.

To conclude, for both risk measures, given our assumptions, the overlapping approach will result in a standard deviation that is  $\sqrt{H}$  times higher as the standard deviation using the square-root-of-time approach.

If the observations  $Y_1, Y_2, \dots$  are not i.i.d., rather, they exhibits serial dependence at the first place, then it further complicates the dependence structure of the overlapped  $H$ -day returns  $Z_1, Z_2, \dots$ . Thus Proposition 3 and Corollary 4 may not hold. In that case, it is necessary to use simulations to compare the two cases, and we do that in the next section.

### 3 Simulation study

While the theoretic results in Section 2 provide guidance as to the asymptotic performance of the estimators on VaR and ES, in typical sample sizes the asymptotics may not yet hold. For that reason, it is of interest to investigate the properties of the estimators for a range of sample sizes that might be encountered in practical applications, and we do that by means of an extensive simulation study.

Though, it would be straightforward to consider other probability levels, for the remainder of this Section, we focus on presenting results from the Basel II and III probabilities, and so we can omit the probability from the notation, unless otherwise indicated. Hence VaR means VaR(99%) and ES means ES(97.5%) below. We report the full set of results for both probabilities in the web appendix at [www.ModelsandRisk.org/VaR-and-ES](http://www.ModelsandRisk.org/VaR-and-ES).

In the results reported below, the sample size of each simulated sample,  $N$ , ranges from 300 to 12,500. For presentation purposes, we denote sample sizes above 300 as years with a year consisting of 250 observations, so a sample size at 12,500 corresponds to 50 years. More comprehensive results are reported in the web appendix.

We consider two fat tailed distributions, the Student- $t$  and Pareto distributions. The results from both are qualitatively similar, so in what follows we focus on the Student- $t$  distributions, leaving the Pareto distributions to the web appendix.

We forecast the two risk measures by HS as in (1). Although the HS

estimator is biased in finite sample, (see e.g. Blom, 1958) and (Danielsson et al., 2015), since our focus is the estimation uncertainty, we do not need to adjust the bias by the methods proposed by Hyndman and Fan (1996).

### 3.1 The number of simulations

The simulations are used not only to obtain estimates of the risk measures, but more importantly the uncertainty of those estimates. This means that in practice we aim to capture the quantiles of the quantiles. Our somewhat ad hoc criteria for the results is that they are accurate for at least three significant digits, and as it turns out it requires at least  $S = 10^7$  simulations. For the largest sample sizes, we are then generating  $S \times N = 10^7 \times 2.5 \times 10^5 = 2.5 \times 10^{12}$  random numbers, and for each sequence need to find a quantile and a mean.

Why is such a large simulation necessary? Taking the VaR measure as an example, from each sample, we obtain one simulated quantity  $\hat{q}_F/q_F - 1$ . Across  $S$  simulated samples, we obtain  $S$  such ratios denoted as  $r_1, r_2, \dots, r_S$ . They are regarded as i.i.d. observations from the distribution of  $\hat{q}_F/q_F$ , denoted as  $F_R$ . Since we intend to obtain the 99% confidence interval of this ratio,  $[F_R^{-1}(0.005), F_R^{-1}(0.995)]$ , we take the  $[0.005S]$ -th and  $[0.995S]$ -th order statistics among  $r_1, \dots, r_S$ ,  $r_{S,[0.005S]}$  and  $r_{S,[0.995S]}$  to be the estimates of the lower and upper bounds respectively. For the lower bound, following

Theorem 2 in Mosteller (1946), we get that as  $S \rightarrow \infty$ ,

$$\sqrt{S} \left( \frac{r_{S,[0.0005S]}}{F_R^{-1}(0.005)} - 1 \right) \xrightarrow{d} N \left( 0, \frac{0.0005 \cdot (1 - 0.0005)}{(F_R^{-1}(0.005))^2 f_R^2(F_R^{-1}(0.005))} \right),$$

where  $f_R$  is the density function of  $F_R$ . Following Proposition 2, the distribution  $F_R$  can be approximated by a normal distribution with a given standard deviation  $\sigma_N$ . Using this approximated distribution, we can explicitly calculate the asymptotic variance above as

$$\sigma_R^2 = \frac{0.005 \cdot (1 - 0.005)}{(\sigma_N \Phi^{-1}(0.005))^2 \left( \frac{1}{\sigma_N} \phi \left( \frac{\sigma_N \Phi^{-1}(0.005)}{\sigma_N} \right) \right)^2} = 3.586.$$

Note that this variance is independent of  $\sigma_N$ . Therefore this result can be applied to any estimator that possesses asymptotic normality.

To ensure that the relative error between our simulated lower bound  $r_{S,[0.0005S]}$  and the actual lower bound  $F_R^{-1}(0.005)$  is less than 0.001 with a confidence level of 95%, the restriction requires a minimum  $S$  such that

$$S \geq \sigma_R^2 * \left( \frac{\Phi^{-1}(0.975)}{0.001} \right)^2 = 1.378 \times 10^7.$$

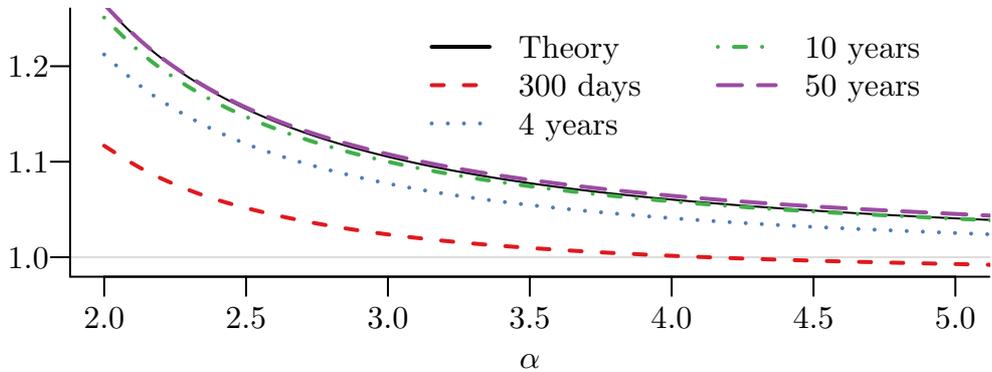
A minimum of  $S = 10^7$  samples is necessary for our simulation study and that is the number of simulated samples we use throughout this section.

### 3.2 Level comparison of the risk measures

The theoretic results in Section 2.1.1 indicate that the relative difference between VaR and ES is small for distributions that do not have very heavy tails, where the difference is inversely related to the tail thickness. We explore this relation by Monte Carlo simulations with a finite sample size. For each given  $\alpha$ , we simulate observations from standard Student-t distribution with degree of freedom  $\nu = \alpha$ , and for each simulated sample, we calculate the ratio between the two estimators  $\hat{e}_F$  and  $\hat{q}_F$ . Such a procedure is repeated  $S$  times for each given sample sizes. We plot the average ratio across different simulated samples with respect to the variation of  $\alpha$ , reported in Figure 1.

Figure 1: Ratio of ES(97.5%) to VaR(99%)

The figure shows the ratio of ES(97.5%) to VaR(99%) for a range of sample sizes. For each given sample size, the number of simulated samples is  $S = 10^7$ .



The solid line shows the theoretical level of the ratio, the  $f(\alpha)$  function in Section 2.1.1, declining towards one as the tails become progressively thinner. The same decline is observed for every sample size. The results for the larger sample sizes, 10 and 50 years, are increasingly close to the theoretic results, and at 50 years, virtually the same.

As the sample size decreases, the relative difference between ES and VaR decreases sharply, especially for the heavier tails. For example, while asymptotic theory suggests that for  $\alpha = 3$ , ES is 11% larger than VaR, at 300 days it is only 3% and 8% at 4 years. At the smallest sample sizes, for tails that are slightly thinner than usual, the ES falls below the VaR.

### 3.3 Estimation accuracy

The asymptotic results in Section 2.1.2 show that ES is estimated more precisely than VaR for relatively thin distributions, and less precisely for the fatter and more typical distributions, with the break even point at  $\alpha = 6$ . Below we investigate this results further for finite samples.

For each given  $\alpha$ , we simulate  $N$  observations from a standard Student-t distribution with degrees of freedom  $\nu = \alpha$ , where  $N$  varies from 300 to 125,000. For each simulated sample, we obtain the two estimates  $\hat{e}_F$  and  $\hat{q}_F$  and calculate the relative estimation error as the ratio between the estimates and their corresponding true values,  $e_F$  and  $q_F$ . Such a procedure is repeated  $S$  times for each given sample size. We then report the mean and standard error of the estimation errors, as well as the 99% empirical confidence interval, corresponding to the 0.5% and 99.5% quantiles from the  $S$  simulated estimation errors, respectively. Table 1 gives the summary information for various sample sizes and tail thicknesses.

We obtain three main results: First, the Monte Carlo results are consistent with the theoretic result in Proposition 2, i.e. ES is estimated with more

Table 1: Comparison of the estimation accuracy

$\alpha$	Sample size	VaR(99%)			ES(97.5%)		
		bias	se	99% conf	bias	se	99% conf
2.5	300 days	1.11	(0.33)	[0.61,2.46]	1.01	(0.38)	[0.54,2.64]
2.5	2 years	1.06	(0.22)	[0.67,1.89]	1.01	(0.29)	[0.61,2.20]
2.5	10 years	1.01	(0.09)	[0.82,1.28]	1.00	(0.13)	[0.78,1.48]
2.5	50 years	1.00	(0.04)	[0.91,1.11]	1.00	(0.06)	[0.89,1.19]
3	300 days	1.09	(0.27)	[0.65,2.16]	1.01	(0.27)	[0.60,2.14]
3	2 years	1.05	(0.19)	[0.70,1.73]	1.00	(0.21)	[0.66,1.82]
3	10 years	1.01	(0.08)	[0.84,1.23]	1.00	(0.09)	[0.82,1.31]
3	50 years	1.00	(0.03)	[0.92,1.09]	1.00	(0.04)	[0.91,1.12]
4	300 days	1.07	(0.21)	[0.69,1.85]	1.00	(0.19)	[0.66,1.73]
4	2 years	1.04	(0.15)	[0.74,1.55]	1.00	(0.15)	[0.72,1.52]
4	10 years	1.01	(0.06)	[0.86,1.19]	1.00	(0.06)	[0.86,1.20]
4	50 years	1.00	(0.03)	[0.93,1.08]	1.00	(0.03)	[0.93,1.08]

Note: For each given  $\alpha$ ,  $N$  observations from a standard Student-t distribution with degree of freedom  $\nu = \alpha$  are simulated. For each simulated sample, the ES and VaR are estimated and then divided by their corresponding true values. The resulting ratio is regarded as the relative estimation error. The table reports the bias (mean), standard error and 0.5% and 99.5% quantiles of these ratios across the  $S = 10^7$  simulated samples. The two quantiles are reported as the lower and upper bounds of the 99% confidence interval. In comparing across the two risk measures, the red values indicates those with the higher standard error.

uncertainty than VaR. This simulation results show that the only exception occurs at the very small sample size combined with a higher  $\alpha$ .

Second, the estimation bias increases as the sample size becomes smaller. This is expected given the HS bias of Blom (1958) and Monte Carlo bias of Danielsson et al. (2015). It also follows that the use of ES will partly offset the HS bias.

Finally, the empirical confidence bounds indicate that the estimation errors are highly positively skewed, especially for the small sample sizes. For

example, at  $N = 300$  and  $\alpha = 2.5$ , the 99% confidence interval for VaR ranges from about 61% to 246% of the true value. Even for an uncommonly large 10-year sample, the confidence bound is  $[0.82, 1.28]$ . For ES(97.5%), the confidence bounds are wider at  $[0.54, 2.64]$  and  $[0.78, 1.48]$ , respectively.

### 3.4 The overlapping approach and the time-scaling approach

The theoretic results in Section 2.2 provided insights into the impact of using overlapping estimation or time-scaling to obtain multi-day holding period risk forecasts. We further extend those results by means of Monte Carlo simulations, both investigating the final example properties but also examining the impact of using dependent data. Below we only report a subset of the results for VaR, as the results for ES were qualitatively similar, with the full results available in the web Appendix.

For each given distribution and  $H$ , we simulate  $N$  daily observations,  $S = 10^7$  times, varying  $N$  from 300 to 12,500 (50 years). For each simulated sample, we estimate the  $H$ -day holding period VaR using both the time-scaling and overlapping date approaches. Similar to the above, we divide the estimates by the true values. Since we estimate the VaR of the  $H$ -day holding period, which is not analytically tractable, we have to rely on pre-simulations with very large data samples to obtain the true values. We consider  $H = 10$  and  $H = 50$ .

### 3.4.1 Data generating process: The i.i.d. case

We start with the i.i.d. case and report the results in Table 2, which is similar to Table 1, with the addition of two columns that show the ratios of the se and the width of the confidence interval, for the overlapping approach over the square-root-of-time approach.

The i.i.d. simulation results are consistent with those predicted by Proposition 3: time-scaling results in better estimation accuracy than the overlapping destination. Both the standard errors and the width of the confidence intervals for the overlapping approach are much higher than that of the time-scaling approach, ranging from 1.3 to 3.9 times larger.

The bias and uncertainty for the overlapping approach first increases and then decreases as the sample size increases, something not observed for the time-scaling approach. We surmise that this happens because in each sample, large daily losses will persist in  $H$ -day losses for  $H$  days, which is a significant fraction of the sample for smaller sample size  $N$ . Consequently, the HS estimators using the overlapping approach may rely on large observations stemming from one daily large loss, which may lead to severely biased estimates, though relatively small estimation uncertainty. As the sample size  $N$  increases further, we move away from the scenario that the persistent large  $H$ -day losses can be regarded as a large fraction of the sample. Therefore, the sample size effect starts to perform as predicted in our theory. Practically, the overlapping approach performs the worst when used for typical sample sizes, such as two years.

Table 2: Impact of overlapping data on VaR: Student-t

(a) H=10

$N$	$\alpha$	overlapping approach			square-root-of-time approach			Ratios of overlap to scaling	
		mean	se	99% conf	mean	se	99% conf	se	range
300 days	2.5	1.01	(0.81)	[0.43,4.48]	1.13	(0.33)	[0.62,2.49]	2.5	2.2
2 years	2.5	1.12	(0.88)	[0.50,5.31]	1.08	(0.23)	[0.68,1.91]	3.8	3.9
10 years	2.5	1.03	(0.21)	[0.70,1.92]	1.03	(0.089)	[0.83,1.29]	2.4	2.7
50 years	2.5	1.00	(0.080)	[0.84,1.26]	1.02	(0.039)	[0.92,1.12]	2.1	2.1
300 days	3.0	1.00	(0.49)	[0.48,3.25]	1.16	(0.29)	[0.69,2.30]	1.7	1.7
2 years	3.0	1.06	(0.54)	[0.56,3.65]	1.12	(0.20)	[0.75,1.84]	2.7	2.8
10 years	3.0	1.01	(0.15)	[0.75,1.61]	1.08	(0.081)	[0.90,1.32]	1.9	2.0
50 years	3.0	1.00	(0.060)	[0.87,1.19]	1.07	(0.035)	[0.98,1.17]	1.7	1.7
300 days	4.0	0.98	(0.29)	[0.53,2.23]	1.17	(0.23)	[0.75,2.02]	1.3	1.3
2 years	4.0	1.02	(0.28)	[0.61,2.32]	1.13	(0.17)	[0.81,1.70]	1.6	1.9
10 years	4.0	1.00	(0.10)	[0.80,1.35]	1.10	(0.068)	[0.94,1.30]	1.5	1.5
50 years	4.0	1.00	(0.043)	[0.90,1.13]	1.09	(0.030)	[1.02,1.17]	1.4	1.5

(b) H=50

$N$	$\alpha$	overlapping approach			square-root-of-time approach			Ratios of overlap to scaling	
		mean	se	99% conf	mean	se	99% conf	se	range
300 days	2.50	0.72	(0.43)	[0.15,2.44]	1.15	(0.34)	[0.63,2.53]	1.3	1.2
2 years	2.50	0.81	(0.46)	[0.27,2.79]	1.09	(0.23)	[0.69,1.94]	2.0	2.0
10 years	2.50	1.08	(0.77)	[0.56,4.55]	1.04	(0.090)	[0.84,1.31]	8.6	8.5
50 years	2.50	1.01	(0.16)	[0.75,1.69]	1.03	(0.039)	[0.94,1.14]	4.1	4.7
300 days	3.00	0.76	(0.32)	[0.17,1.95]	1.20	(0.30)	[0.71,2.37]	1.1	1.1
2 years	3.00	0.84	(0.32)	[0.30,2.12]	1.15	(0.21)	[0.77,1.89]	1.5	1.6
10 years	3.00	1.02	(0.39)	[0.61,2.89]	1.11	(0.083)	[0.92,1.35]	4.7	5.3
50 years	3.00	1.00	(0.11)	[0.80,1.39]	1.10	(0.036)	[1.01,1.20]	3.1	3.1
300 days	4.00	0.78	(0.27)	[0.18,1.65]	1.20	(0.24)	[0.77,2.08]	1.1	1.1
2 years	4.00	0.86	(0.25)	[0.33,1.70]	1.16	(0.17)	[0.83,1.74]	1.5	1.5
10 years	4.00	0.99	(0.19)	[0.65,1.74]	1.13	(0.070)	[0.97,1.33]	2.7	3.0
50 years	4.00	1.00	(0.076)	[0.83,1.23]	1.12	(0.031)	[1.05,1.21]	2.5	2.5

Note: For each given  $\alpha$  and holding period  $H$ ,  $N$  daily observations from standard Student-t distribution with degree of freedom  $\nu = \alpha$  are simulated with a year consisting of 250 days. For each simulated sample, the VaR of  $H$ -day holding period is estimated using the two strategies in Section 2.2 separately, and then divided by the corresponding true value obtained from pre-simulations. The resulting ratio is regarded as the relative estimation error. The table reports the mean, standard error and 0.5% and 99.5% quantiles of these ratios across  $S$  simulated samples with  $S = 10^7$ . The two quantiles are reported as the lower and upper bounds of the 99% confidence interval. The last two columns show the ratios of the se and the width of the confidence interval, for the overlapping approach over the square-root-of-time approach.

### 3.4.2 Data generating process: Dependent data

The overlapping approach induces serial dependence and is therefore likely to be especially sensitive to the inherent dependence of the data. We therefore also explore the impact of simulating from dependent data, using a specification that both captures the fat tails and dependence. There are many different ways one could specify such a model. A commonly used specification would be a normal GARCH model, but such a model would not adequately capture the tails, (see e.g. Sun and Zhou, 2014) and we therefore opted for a GARCH model with Student- $t$  innovations. We parameterized the simulation model by estimating the same specification for a number of stocks and picking a typical set of parameters. In particular:

$$\begin{cases} X_t = \sigma_t \varepsilon_t; \\ \sigma_t^2 = 0.01 + 0.94\sigma_{t-1}^2 + 0.04X_{t-1}^2; \end{cases} \quad (2)$$

where  $\varepsilon_t$  are i.i.d. innovation terms following a standardized Student- $t$  distribution with degree of freedom 6 and unit variance.

Table 3 reports the result based on daily observations generated from the GARCH model. Notice that due to the serial dependence in the GARCH model, our theoretical result in Proposition 3 may not hold. Therefore, we have to rely on the simulation result for comparing the two approaches.

Compared to the i.i.d. case, the time-scaling approach results in even lower standard errors than the overlapping approach. In addition, there are two important differences between the i.i.d. and dependent cases for the overlapping approach. First, in the dependent case, the standard errors and biases decrease as  $N$  increases. Second, for  $H = 50$ , and  $N$  less than 10

Table 3: Impact of overlapping data on VaR, t-GARCH(0.01, 0.04, 0.94, 6.0)

(a) H=10

$N$	overlapping approach			square-root-of-time approach			Ratios of overlap to scaling	
	mean	se	99% conf	mean	se	99% conf	se	range
300 days	1.01	(0.33)	[0.49,2.38]	1.14	(0.29)	[0.68,2.33]	1.1	1.1
2 years	1.02	(0.29)	[0.57,2.28]	1.11	(0.22)	[0.72,2.00]	1.3	1.3
10 years	1.01	(0.14)	[0.75,1.52]	1.06	(0.100)	[0.86,1.40]	1.4	1.4
50 years	1.00	(0.059)	[0.87,1.18]	1.05	(0.044)	[0.95,1.18]	1.3	1.3

(b) H=50

$N$	overlapping approach			square-root-of-time approach			Ratios of overlap to scaling	
	mean	se	99% conf	mean	se	99% conf	se	range
300 days	0.78	(0.32)	[0.17,2.00]	1.16	(0.29)	[0.69,2.37]	1.1	1.1
2 years	0.85	(0.31)	[0.30,2.05]	1.12	(0.23)	[0.74,2.02]	1.3	1.4
10 years	0.99	(0.24)	[0.60,2.00]	1.08	(0.10)	[0.87,1.42]	2.4	2.5
50 years	1.00	(0.10)	[0.79,1.33]	1.07	(0.044)	[0.96,1.20]	2.3	2.3

Note: For each holding period  $H$ ,  $N$  daily observations from the GARCH model (2) are simulated. For each simulated sample, the VaR of  $H$ -day holding period is estimated using the two strategies in Section 2.2 separately, and then divided by the corresponding true value obtained from pre-simulations. The resulting ratio is regarded as the relative estimation error. The table reports the mean standard error and 0.5% and 99.5% quantiles of these ratios across  $S$  simulated samples with  $S = 10^7$ . The two quantiles are reported as the lower and upper bounds of the 99% confidence interval. The last two columns show the ratios of the se and the width of the confidence interval, for the overlapping approach over the square-root-of-time approach.

years, there is a downward bias, i.e. the estimates are lower than the true value. The bias can be around 20% for low values of  $N$ .

The first difference provides some support for using the overlapping approach for dependent data, even though the time-scaling approach still performs better in terms of estimation accuracy. This benefit is counteracted by the second observation, where, for example, from a viewpoint of a prudential regulator, the lower bound of the confidence interval based on overlapping approach is much lower than that based on time-scaling approach.

## 4 Empirical results

While the theoretic and simulation results above provide a clear picture of the relative properties of various risk measures, they were obtained under particular distributional assumptions. To augment those results, we also employ observed returns from CRSP. The downside is that since we do not know the true value of the risk measures, we cannot directly validate the estimation uncertainty, but can be approximated by means of a block bootstrapping procedure.<sup>2</sup>

Our data sets consist of daily returns on all stock prices traded on NASDAQ, NYSE or AMSE from 1926 to 2014. We removed illiquid stocks<sup>3</sup> as well as those with less than 650 observations.<sup>4</sup> This filtering procedure results in 7,686 stocks. For each stock, we split the time series into non-overlapping samples with sample sizes  $N$  and then pool all samples from different stocks together. The sample sizes in this analysis are  $N = 600, 1000$  and  $5000$  resulting in 34,244, 20,097 and 2,503 samples for each of the three sample sizes, respectively.

Similar to the simulation analysis, the probability levels are 99% for VaR and 97.5% for ES. For expositional expediency, we therefore drop a reference

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<sup>2</sup>The block size needs to be large enough to capture the inherent dependence in the data, and we opted for 200 days.

<sup>3</sup>The liquidity criterion is related to the sample splitting procedure. A non-overlapping sample of one stock is included in our analysis if on the first day of the sample the stock has a share price above 5\$ and market capitalization is higher than the 10% quantile of the market capitalization of all stocks traded on NYSE on that day.

<sup>4</sup>The minimum sample size is determined at  $N + H$ , where  $N$  is the minimum sample size 600 and  $H$  is the longest liquidity horizon in our analysis 50. The reason to choose the minimum sample size at  $N = 600$  instead of 500 is due to the block bootstrapping procedure: the sample size is required to be a multiple of the block size, 200.

to the probabilities from the notation.

In line with the theoretic and simulation analysis, the real data study consists of three parts: a comparison of the levels of VaR and ES, the relative estimation uncertainty between VaR and ES and overlapping and time-scaling approaches.

#### 4.1 Level comparison

To compare the levels of VaR and ES, we calculate the ratio  $ES/VaR$  for each sample and report the cross-sectional mean, median and standard error. In addition, we report the fraction of samples with a ratio above one in the row ratio. Finally, we do a t-test to test the mean equaling one across all samples, and report the  $p$ -value in the row  $p$ -value. All results are in Column 4 of Table 4.

We observe that the level of ES is slightly but statistically significantly higher than that of VaR for each of the sample sizes respectively. However, the economic significance is rather weak, especially for the smaller and more typical sample sizes such as  $N = 600$ .

Following the simulation results in Figure 1, the ratio for large sample sizes is close to its theoretical value  $f(\alpha)$ . By taking the mean ratio 1.06 under  $N = 5,000$ , we invert the relation  $f(\alpha) = 1.06$  to get that  $\alpha^* = 4.01$ . This result is in line with the estimated tail index of stock returns in literature, see, e.g. Jansen and de Vries (1991).

## 4.2 Estimation accuracy

In order to compare the estimation uncertainty of VaR and ES, we start with calculating the *variation coefficient ratio* (VCR) between ES and VaR, for each sample. The VCR between two risk measures  $\varphi_1$  and  $\varphi_2$  is defined as

$$\text{VCR}(\varphi_1, \varphi_2) = \frac{\sigma(\varphi_1)/\hat{\varphi}_1}{\sigma(\varphi_2)/\hat{\varphi}_2},$$

where  $\hat{\varphi}_i$  is the point estimate for  $\varphi_i$  and  $\sigma(\varphi_i)$  is the standard error of the estimation obtained from block bootstrapping,  $i = 1, 2$ .

The block bootstrapping procedure is as follows. We randomly draw a number of blocks consisting of consecutive observations with a block size  $B = 200$  from the given sample. With  $N/B$  blocks, we construct a bootstrapped sample with sample size  $N$ . For each bootstrapped sample  $j$ , we get the point estimate of  $\varphi_i$  as  $\hat{\varphi}_i^{(j)}$ , where  $j = 1, 2, \dots, K$ . Here the number of replication  $K = 5,000$ . Then we calculate the standard error  $\sigma(\varphi_i)$  as the sample standard deviation among the  $K$  bootstrapped estimates.

From the theoretic analysis of i.i.d. data, one would expect that the VCR (ES, VaR) exceeds one. However, the results in Column 5 in Table 4 show that the VCR (ES, VaR) exceeds one only for the largest sample size  $N = 5,000$  so the empirical results only partially support the theoretic results. This is comparable with some exceptional results in the simulation study, see Table 1, the last panel with  $\alpha = 4$ .

To further explore the economic impact of estimation uncertainty, we compare the lower bound of the 99% confidence interval when estimating the

two risk measures. For each given sample, using the bootstrapping procedure, we take the  $0.005K$ -th and  $0.995K$ -th quantiles among the  $K$  bootstrapped estimates of VaR to construct the 99% confidence interval for the VaR estimate. We focus on the lower bound and denote it as  $l(\text{VaR})$ . Similarly, we obtain  $l(\text{ES})$ . Then, we calculate the ratio between the standardized lower bounds as

$$Q(\text{VaR}, \text{ES}) = \frac{l(\text{VaR})/\widehat{\text{VaR}}}{l(\text{ES})/\widehat{\text{ES}}}$$

We report the outcome based on these ratios in the 5th column of Table 4. The lower bound of VaR is significantly higher than that of ES across all sample sizes, in line with our simulation results in Table 1.

### 4.3 The overlapping and time-scaling approaches

Finally, we compare the overlapping approach and the time-scaling approach, in particular the square-root-of-time approach. The notation VCR (VaR  $H10$ ) is the VCR between the VaR measures using the overlapping approach and square-root-of-time approach, and similarly for ES and  $H = 50$ . The results are reported in the last four columns of Table 4.

We find strong evidence that all four VCRs are significantly above one. These results are in line with our qualitative conclusion drawn from theoretical analysis and simulations. The average VCR for  $H = 10$  is below  $\sqrt{10}$  and that for  $H = 50$  is below  $\sqrt{50}$ . Therefore, the empirical VCR is lower than the predicted VCR when assuming independence. Nevertheless, they are close to the simulation results when the DGP is a t-GARCH process,

see Table 3. Hence, we conclude both from simulation and real data analysis that the serial dependence leads to VCR that are lower than those derived from the independent case.

## 5 Concluding remarks

In this paper we focus on three key issues in risk analysis: the choice of risk measures, the aggregation method when considering longer holding period and the number of observations needed for accurate risk forecast.

First, by comparing the most commonly used risk measures, the VaR and ES, we conclude that the theoretic superiority of ES over VaR comes at the cost of higher estimation error for ES. For many end-users, using typical sample sizes of a few hundred to a few thousand observations, this may well tip the advantage in favor of VaR.

Second, the time-scaling approach is much more accurate than the overlapping approach and hence to be preferred. This certainly holds for i.i.d. data and with the dependent structure considered here. Indeed, there seems to be little reason to use the overlapping approach when forecasting risk.

Finally, both ES and VaR are highly sensitive to the sample size. We need half a century of daily data for the estimators to reach their asymptotic properties, with the uncertainty increasing rapidly with lower sample sizes. For the smaller sample sizes, below a few thousand days, the uncertainty becomes considerable, and at 500 or less very little signal remains. Consider the case of typically thick tails ( $\alpha = 3$ ) and a 500 day sample size. In this

Table 4: Empirical analysis with the CRSP data

$N$	Number of Samples		Level	ES and VaR uncertainty		VCR holding period comparisons			
			ES/VaR	VCR (ES, VaR)	Q(VaR, ES)	VaR $H10$	VaR $H50$	ES $H10$	ES $H50$
600	34,244	mean	1.03	0.91	1.02	1.39	2.28	1.41	2.53
		median	1.01	0.86	1.01	1.16	1.78	1.24	2.04
		se	0.09	0.39	0.11	1.01	2.49	0.73	2.48
		ratio	0.57	0.29	0.55	0.61	0.85	0.70	0.93
		$p$ -value	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1000	20,097	mean	1.04	0.96	1.01	1.40	1.90	1.33	1.97
		median	1.03	0.92	1.01	1.22	1.63	1.24	1.73
		se	0.08	0.29	0.09	0.80	1.06	0.53	0.96
		ratio	0.71	0.34	0.54	0.67	0.88	0.73	0.93
		$p$ -value	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5000	2,503	mean	1.06	1.01	1.01	1.43	2.03	1.35	1.85
		median	1.06	0.98	1.01	1.35	1.83	1.31	1.75
		se	0.04	0.18	0.04	0.49	0.89	0.33	0.64
		ratio	0.97	0.45	0.58	0.83	0.95	0.87	0.96
		$p$ -value	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Note: The table shows the empirical results using daily returns on all liquid traded stocks on NASDAQ, NYSE or AMSE from 1926 to 2014. Each stock return series is split into non-overlapping samples with sample sizes  $N = 600, 1,000, 5,000$ . All samples from different stocks are pooled together. A sample is included if on the first day of the sample the stock has a share price above 5\$ and market capitalization higher than the 10%th quantile of the market capitalization of all stocks traded on NYSE on that day. The number of samples are reported in column 2. Column 4 reports the summary statistics of the ratios ES/VaR across all samples. The row “ratio” reports the fraction of samples with a ratio above one. The row “ $p$ -value” reports the  $p$ -value of a t-test that the mean equals to one. The empirical results for the variation coefficient ratio (VCR) between ES and VaR are reported in Column 5. For the calculation of the VCR, a block bootstrapping procedure is employed with details in Section 4. Column 5 reports the empirical results based on the ratio Q(VaR, ES), which measures the relative difference in the lower bound of the confidence intervals for VaR and ES with the calculation details in Section 4. The last four columns report the empirical results based on the VCR between the a given risk measure and liquidity horizon using the overlapping approach and square-root-of-time approach.

case, the 99% confidence interval around the true value of one is  $[0.70, 1.73]$  for VaR(99%) and  $[0.66, 1.82]$  for ES(97.5%).

As a side result, we also observed estimation biases using the HS method, expected given Blom (1958). Nevertheless, sometimes the estimation bias can work in the “right” direction from the regulators’ perspective: an upwards bias may lead to a relatively high value on the lower bound and consequently less room for under reporting. Therefore, a full discussion on estimation uncertainty should take into account both bias and variance.

These results have a number of conclusions for both a practical use of the risk forecasts and the impact on financial regulations.

First, in the latest Basel III market risk proposals, the Basel committee suggests replacing 99% VaR with 97.5% ES. Our results indicate that this will lead to less accurate risk forecasts. If the regulators are concerned by precision, VaR is preferred. In addition, in some cases the 99% VaR is higher than the 97.5% ES, meaning that the move from Basel II to Basel III may result in lower risk forecasts. This is especially likely for the smaller sample sizes most likely to be encountered in practice.

However, there are reasons beyond our study for which ES should be preferred. For instance, ES is harder to manipulate than VaR and therefore might be preferred even if it is less accurate. A financial institution may manipulate the risk forecast by picking a particular estimation method. However, there is no guarantee that an estimation method that delivers a favorable result today will deliver similar favorable results in the future. Instead, manipulation is much likely to happen by picking trades that place

assets on the lower boundary of the confidence interval, something very easy to do while being virtually undetectable. Because VaR is only one point on the distribution of profit and loss and ES captures the entire tail from that point, it is harder to implement such manipulation on ES than if VaR is used.

Second, the high estimation uncertainty for risk measures provides one explanation for why violation ratios in back testing so often deviate from the expected values by large amounts. When looking at the confidence intervals for risk forecasts in our study, the lower bounds give banks some scope for deliberately underreporting risk, perhaps by cherry picking trades known to be on the lower edge. Since the lower bounds can be regarded as a reasonable estimate of the risk metric within a given confidence level, the violation ratio based on the lower bound should also be regarded as acceptable at the same confidence level. Consequently, if the lower bound is reported as an acceptable estimate of the risk, the Bernoulli distribution for violation will have a higher probability of violation. Coupled with the well-known small sample problem in the Bernoulli distribution in back testing, the actual realized violation ratio may largely deviate from the expected value as the probability level used in the risk metric. This makes the backtesting procedure on reported risk measures challenging.

Finally, the results in our study indicate that it is not advisable for neither financial institutions nor the regulators to rely solely on point estimates of risk forecasts. It is important to also report the confidence bounds. Furthermore, given the highly asymmetric nature of these bounds, the actual bounds should be reported, rather than reporting the standard error only.

# Appendix

## Proof of Proposition 1.

Recall that  $F$  is a heavy-tailed distribution with tail index  $\alpha$ . Danielsson et al. (2006) showed that if  $\alpha > 1$

$$\lim_{s \rightarrow 0} \frac{e_F(1-s)}{q_F(1-s)} = \frac{\alpha}{\alpha-1}. \quad (3)$$

In addition, from the regular variation condition, we get that

$$\lim_{s \rightarrow 0} \frac{q_F(1-cs)}{q_F(1-s)} = (2.5)^{-1/\alpha}. \quad (4)$$

The proposition is proved by combining Eq. (3) and (4). ■

## Proof of Proposition 2.

Under the conditions in the proposition, Theorem 2.4.8 in de Haan and Ferreira (2006) showed that there exists a proper probability space with Brownian motions  $\{W_N(s)\}_{s \geq 0}$  such that as  $N \rightarrow \infty$ ,

$$\left| \sqrt{k} \left( \frac{X_{N, N-[ks]}}{U(N/k)} - s^{-1/\alpha} \right) - \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1} W_N(s) - \sqrt{k} A(N/k) s^{-\frac{1}{\alpha}} \frac{s^{-\rho} - 1}{\rho} \right| \xrightarrow{P} 0 \quad (5)$$

holds uniformly for all  $0 < s \leq 1$ .

By taking  $s = 1$ , the first statement on  $\hat{q}_F(1 - k/N)$  follows immediately. To prove the second statement on  $\hat{e}_F(1 - k/N)$ , we apply the integral for

$s \in (0, 1]$  to (5) and obtain that as  $N \rightarrow \infty$

$$\sqrt{k} \left( \frac{\hat{e}_F(1 - k/N)}{U(N/k)} - \frac{1}{1 - 1/\alpha} \right) - \int_0^1 \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1} W_N(s) ds - \lambda \frac{1}{(1 - \rho)(1 - 1/\alpha - \rho)} \xrightarrow{P} 0.$$

Notice that it is necessary to have  $\alpha > 2$  to guarantee the integrability of  $\int_0^1 \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1} W_N(s) ds$ .

Similarly, from the inequality (2.3.23) in de Haan and Ferreira (2006), we get that for any  $\varepsilon > 0$ , with sufficiently large  $N$ ,

$$\left| \sqrt{k} \left( \frac{U(N/ks)}{U(N/k)} - s^{-1/\alpha} \right) - \sqrt{k} A(N/k) s^{-\frac{1}{\alpha}} \frac{s^{-\rho} - 1}{\rho} \right| \leq \varepsilon \sqrt{k} A(N/k) s^{-1/\alpha - \rho - \varepsilon},$$

holds for all  $0 < s \leq 1$ . With a small  $\varepsilon$  such that  $1/\alpha + \rho + \varepsilon < 1$ , we can take integral for  $s \in (0, 1]$  on both sides and obtain that as  $N \rightarrow \infty$ ,

$$\sqrt{k} \left( \frac{e_F(1 - k/N)}{U(N/k)} - \frac{1}{1 - 1/\alpha} \right) \rightarrow \lambda \frac{1}{(1 - \rho)(1 - 1/\alpha - \rho)}.$$

Therefore, by comparing the asymptotics of  $\frac{\hat{e}_F(1 - k/N)}{U(N/k)}$  and  $\frac{e_F(1 - k/N)}{U(N/k)}$ , we get that

$$\sqrt{k} \left( \frac{\hat{e}_F(1 - k/N)}{e_F(1 - k/N)} - 1 \right) \xrightarrow{d} \frac{\alpha - 1}{\alpha^2} \int_0^1 s^{-\frac{1}{\alpha}-1} W(s) ds.$$

The proof is finished by verifying the variance of the limit distribution as

follows.

$$\begin{aligned}
\text{Var} \left( \frac{\alpha - 1}{\alpha^2} \int_0^1 s^{-\frac{1}{\alpha}-1} W(s) ds \right) &= \frac{(\alpha - 1)^2}{\alpha^4} \int_0^1 ds \int_0^1 dt \left( s^{-\frac{1}{\alpha}-1} t^{-\frac{1}{\alpha}-1} \min(s, t) \right) \\
&= \frac{2(\alpha - 1)^2}{\alpha^4} \int_0^1 dt \left( t^{-\frac{1}{\alpha}-1} \int_0^t s^{-\frac{1}{\alpha}} ds \right) \\
&= \frac{2(\alpha - 1)}{\alpha^3} \int_0^1 t^{-\frac{2}{\alpha}} dt \\
&= \frac{2(\alpha - 1)}{\alpha^2(\alpha - 2)}.
\end{aligned}$$

■

### Proof of Proposition 3.

Proposition 2 was proved based on the limit relation (5). We refer to a similar relation based on dependent data, see Theorem 2.1 in Drees (2003). There exists a proper probability space with Gaussian processes  $\{B_N(s)\}_{s \geq 0}$  such that

$$\left| \sqrt{k} \left( \frac{X_{N, N-[ks]}}{U(N/k)} - s^{-1/\alpha} \right) - \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1} B_N(s) - \sqrt{k} A(N/k) s^{-\frac{1}{\alpha}} \frac{s^{-\rho} - 1}{\rho} \right| \xrightarrow{P} 0 \quad (6)$$

holds uniformly for all  $0 < s \leq 1$ , as  $N \rightarrow \infty$ . Here the Gaussian processes  $\{B_N(s)\}_{s \geq 0}$  has a covariance function  $c(x, y) := \text{Cov}(B_N(x), B_N(y))$  determined by the dependence structure as follows. Denote  $c_m(x, y)$  as the tail dependence function between  $X_1$  and  $X_{1+m}$  as

$$\lim_{t \rightarrow \infty} t \Pr(X_1 > U(t/x), X_{1+m} > U(t/y)) = c_m(x, y).$$

Then

$$c(x, y) = \min(x, y) + \sum_{m=1}^{+\infty} (c_m(x, y) + c_m(y, x)).$$

We calculate the specific  $c$  function under the moving average structure  $X_i = \sum_{j=1}^H Y_{i+j-1}$ . It is clear that  $c_m(x, y) = 0$  for  $m \geq H$ . Next, for  $1 \leq m < H$ , we have that

$$\begin{aligned} c_m(x, y) &= \lim_{t \rightarrow \infty} t \Pr(X_1 > U(t/x), X_{1+m} > U(t/y)) \\ &= \lim_{t \rightarrow \infty} t \Pr\left(\sum_{j=m+1}^H Y_j > \max(U(t/x), U(t/y))\right) \\ &= \lim_{t \rightarrow \infty} t(H-m) \Pr(Y_j > U(t/\min(x, y))) \\ &= \frac{H-m}{H} \min(x, y). \end{aligned}$$

Consequently,

$$c(x, y) = \min(x, y) + \min(x, y) \cdot 2 \sum_{m=1}^{H-1} \frac{H-m}{H} = H \min(x, y).$$

The covariance function of  $B_N(s)$  indicates that we can write  $B_N(s) = \sqrt{H}W_N(s)$ , where  $W_N$  is a standard Brownian motion. The proposition is thus proved by following similar steps as in the proof of Proposition 2. ■

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