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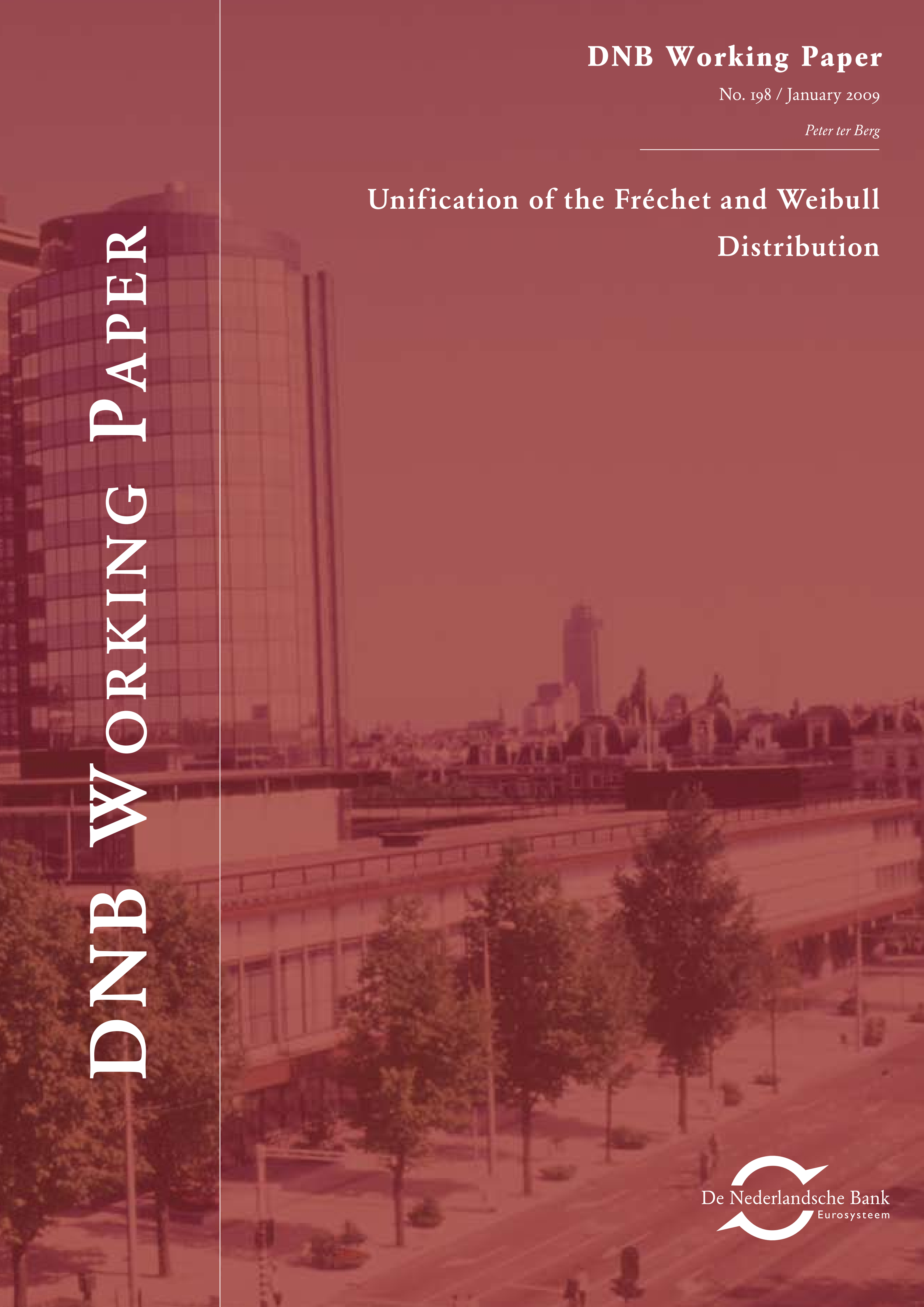

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Unification of the Fréchet and Weibull Distribution

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* Views expressed are those of the author and do not necessarily reflect official positions of De Nederlandsche Bank.

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UNIFICATION OF THE FRÉCHET AND WEIBULL DISTRIBUTION

By PETER TER BERG

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ABSTRACT: Well-known results for the Fréchet and Weibull distribution are streamlined using a unifying parametrisation. Expected values for order statistics follow through a fractional matrix power and the likelihood surface in case of a loglinear specification for the scale parameter is shown to have just two stationary points.

KEYWORDS: Fréchet, Weibull, Gumbel, incomplete Gamma, Lorenz curve, order statistics, fractional matrix power, maximum likelihood.

JEL Classification: C13, C16

RUNNING HEAD: Fréchet and Weibull

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1. INTRODUCTION

Both the Fréchet and the Weibull probability distribution are power transformations of an exponential distributed variate. More specifically, the reciprocal of a Weibull variate is a Fréchet variate. Yet, historically these distributions developed separately. This paper will show that a reciprocal use of the traditional shape parameter of these distributions unifies both distributions. This particular parametrisation makes the distribution and density look less appealing but all other standard formulae will be simplified considerably.

The outline of this paper is as follows. After rephrasing the various standard properties of the Fréchet and Weibull distribution in a more concise format, we will display a collection of expected values for min-max modified power exponential variates. This needs the incomplete Gamma function, which also is useful for the Lorenz curve. Then, the mean of order statistics is expressed in terms of a fractional power of a triangular matrix, leading to a stable computational procedure. Finally we analyse the likelihood function in case of a loglinear specification for the scale parameter. This reveals that there are just two stationary solutions, one for the Weibull and one for the Fréchet case.

All results in this paper are easy to verify or derive once known. The only mathematical exotics could be the incomplete gamma and polygamma functions, and the fractional matrix power. To avoid any confusion, it should be said that this note has nothing or little to do with the three-parameter generalized extreme value distribution that comprises the Gumbel and shifted Fréchet and Weibull distribution. Moreover, it should be acknowledged that WEIBULL (1961) also used a reciprocal shape parameter, without any explanation but without any follow up by others either.

2. Power Exponential Distribution

If a positive variate T is such that $(T/\theta)^{\alpha}$ has an exponential distribution with mean 1, the probability distribution for T depends on the sign of α

$$F(t\mid\alpha,\theta)=\exp\left(-\left[\frac{t}{\theta}\right]^{\alpha}\right)\qquad \alpha<0 \qquad \text{Fréchet}$$

$$=1-\exp\left(-\left[\frac{t}{\theta}\right]^{\alpha}\right)\qquad \alpha>0 \qquad \text{Weibull}$$

Here θ is a scale parameter measured in the same units as the variate T and α is a shape parameter. The density function follows simply as

$$f(t\mid\alpha,\theta)=\frac{|\alpha|}{\theta}\left[\frac{t}{\theta}\right]^{\alpha-1}\exp\left(-\left[\frac{t}{\theta}\right]^{\alpha}\right)\qquad t,\theta>0 \qquad \alpha\neq 0$$

It will be useful to replace the shape parameter α by a shape parameter $\tau=\alpha^{-1}$ and demote the role of α as an auxiliary one. Indeed, the deliberate use of τ instead of α forms the key for the unification of the Fréchet and Weibull distribution as a power exponential distribution. The density becomes:

$$f(t\mid\tau,\theta)=\frac{1}{\theta\,|\tau|}\left[\frac{t}{\theta}\right]^{1/\tau-1}\exp\left(-\left[\frac{t}{\theta}\right]^{1/\tau}\right)\qquad t,\theta>0\,;\quad -\infty<\tau<\infty$$

When τ approaches 0 we get a degenerate distribution at $t=\theta$.
Setting $p=F(t)$, the quantile function follows as:

$$F^{-1}(p)=\theta\left(-\ln p\right)^{\tau}\qquad \tau<0$$

$$=\theta\left(-\ln(1-p)\right)^{\tau}\qquad \tau>0$$

The following characteristics, as function of τ, are simple and are valid irrespective whether τ is negative or positive. They remain valid when $\tau=0$. We display the mode M, median m and mean μ as:

$$M=\theta(1-\tau)^{\tau}\qquad \tau<1$$

$$m=\theta\left(\ln 2\right)^{\tau}$$

$$\mu=\theta\,\Gamma(1+\tau)\qquad \tau>-1$$

These are smooth functions in τ. Putting $\theta = 1$, the graphs of the logarithm of *M*, *m*, and μ are concave, linear and convex in τ respectively. These curves cross at $\tau = 0$ and at points near $\tau = 0.3$. More specifically, following GROENEVELD (1986), we may derive:

$$m = \mu \qquad \tau = 0.2907365$$
$$M = \mu \qquad \tau = 0.3018896$$
$$M = m \qquad \tau = 0.3068528 = 1 - \ln(2)$$

The traditional ordering $M < m < \mu$ holds for $\tau < 0$ and $\tau > 1 - \ln(2)$. As a result the density is skewed to the left for small positive values of τ. It becomes rather symmetric for values of τ near 0.3 and skewed to the right for larger values.
The points of inflection are smooth functions in τ too:

$$\theta\left(\frac{3(1-\tau)+\sqrt{(1-\tau)(5-\tau)}}{2}\right)^{\tau} \qquad \tau < 1$$
$$\theta\left(\frac{3(1-\tau)-\sqrt{(1-\tau)(5-\tau)}}{2}\right)^{\tau} \qquad \tau < ½$$

However, they switch their interpretation as a left or right point of inflection when crossing $\tau = 0$. The Fréchet-case τ<0 has high contact at the origin and two points of inflection although they tend to 0 when $\tau \to -\infty$. Moreover the uppertail is of the Pareto-type with Pareto constant $-\alpha$. For the Weibull case with 0<τ<1 there is a light upper tail with one or two points of inflection. When τ>1 the density is monotone decreasing and subexponential. A movie of the changing shape behavior of the power exponential density can be obtained by putting the median $m = 1$, that is $\theta = (\ln 2)^{-\tau}$ and taking τ as movie-time.

When *T* follows a power exponential distribution with parameters (τ, θ), a power transform T^s with *s* integer, fractional or negative follows a power exponential distribution with adjusted parameters given by $(s\tau, \theta^s)$. This releases us to consider values of *s* other than 1. We make an exception for the variance

$$V(T) = \theta^2\Gamma(1+2\tau) - \mu^2 = \mu^2\left\{\binom{2\tau}{\tau} - 1\right\} \qquad \tau > -½$$

The coefficient of variation follows as the square root of the term between braces. On the interval (0,1.4) this coefficient of variation is close to τ as figure 1 shows

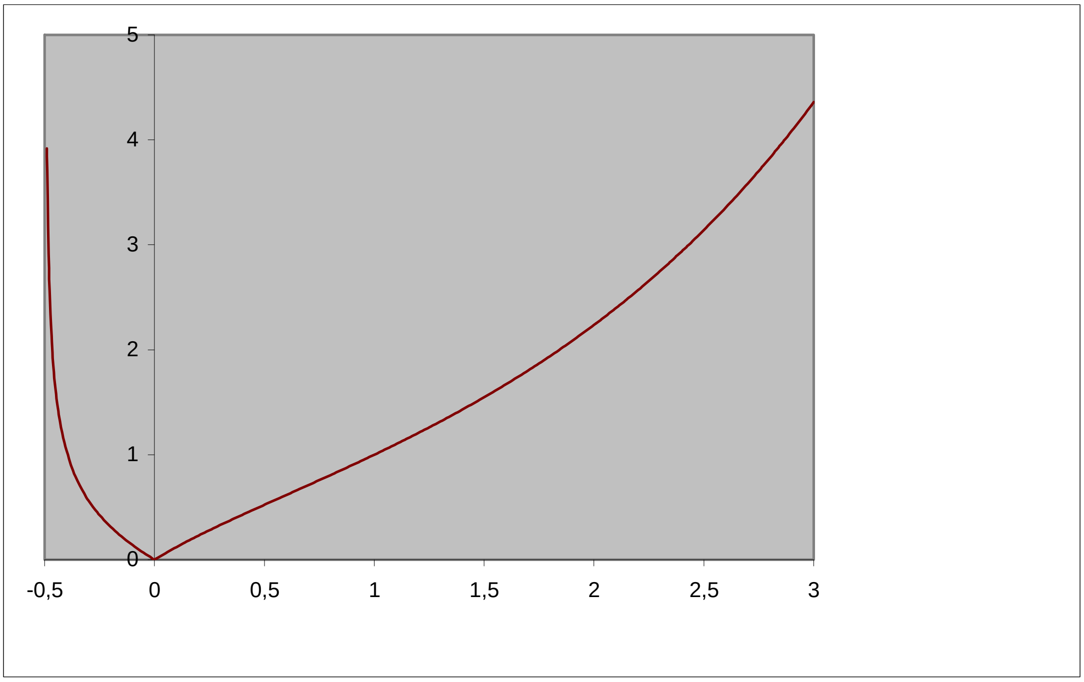

Figure 1: Coefficient of Variation as function of τ

Finally we mention that the largest (smallest) order statistic from a Fréchet (Weibull) sample of size n follows a Fréchet (Weibull) distribution with scale parameter adjusted to $\theta n^{-\tau}$.

3. Results Using Incomplete Gamma Functions

The following results are less nice than the results presented in Section 2 in the sense that the Fréchet and Weibull case do not boil down to single expressions. Yet, the use of τ instead of α as well as some pursuit for simplification, generates convenient results. We will need the incomplete gamma functions $\Gamma(a,x)$, $Q(a,x)$ and $P(a,x)$ defined as:

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} \mathrm{d}t \qquad x > 0, \quad a \in \Re$$
$$Q(a,x) = \Gamma(a,x) / \Gamma(a) \qquad x \geq 0, \quad a > 0$$
$$P(a,x) = 1 - Q(a,x)$$

A useful recursion is $\Gamma(1+a,x) = a\Gamma(a,x) + x^a e^{-x}$.

For computational purposes the continued fraction

$$\Gamma(a,x) = x^a e^{-x} \cdot \frac{1}{x+}\ \frac{1-a}{1+}\ \frac{1}{x+}\ \frac{2-a}{1+}\ \frac{2}{x+}\ \frac{3-a}{1+}\ \frac{3}{x+}\ \frac{4-a}{1+}\ \frac{4}{x+\cdots}$$

will do for most cases, especially for large x and x larger than a. For further numerical strategies we refer to DIDONATO and MORRIS (1986). With these incomplete Gamma functions we may display various concise expressions for such quantities as limited expected values and mean excess functions. To this end we introduce a barrier value b and its dimensionless transform $\tilde{b} = (b/\theta)^\alpha$. The results in Table 1, where $\mu = \theta\Gamma(1+\tau)$ are now straightforward to verify.

TABLE 1: EXPECTED VALUES MIN-MAX MODIFIED POWER-EXPONENTIAL VARIATES

variate	$-1 < \tau < 0$ [a]	$0 < \tau$
$T \cdot I_{\{T \le b\}}$	$\theta\Gamma(1+\tau,\tilde{b}) < b$	$\mu P(1+\tau,\tilde{b})$
$T \cdot I_{\{T > b\}}$	$\mu P(1+\tau,\tilde{b})$	$\theta\Gamma(1+\tau,\tilde{b}) = \mu Q(1+\tau,\tilde{b})$
$\min(T,b)$	$b + \theta\tau\Gamma(\tau,\tilde{b}) < b$	$\mu - \theta\tau\Gamma(\tau,\tilde{b}) = \mu P(\tau,\tilde{b})$
$\max(T,b)$	$\mu - \theta\tau\Gamma(\tau,\tilde{b})$	$b + \theta\tau\Gamma(\tau,\tilde{b}) = b + \mu Q(\tau,\tilde{b})$
$T \mid T < b$	$\theta\Gamma(1+\tau,\tilde{b})\exp(\tilde{b}) < b$	$\dfrac{\mu P(1+\tau,\tilde{b})}{1-\exp(-\tilde{b})}$
$T \mid T > b$	$\dfrac{\mu P(1+\tau,\tilde{b})}{1-\exp(-\tilde{b})}$	$\theta\Gamma(1+\tau,\tilde{b})\exp(\tilde{b})$
$\max(T-b,0)$	$\mu - \theta\tau\Gamma(\tau,\tilde{b}) - b$	$\theta\tau\Gamma(\tau,\tilde{b}) = \mu Q(\tau,\tilde{b})$
$(T-b) \mid T > b$	$\dfrac{\mu P(1+\tau,\tilde{b})}{1-\exp(-\tilde{b})} - b$	$\theta\tau\Gamma(\tau,\tilde{b})\exp(\tilde{b})$

[a] The three entries with the extension $< b$ are also valid for $\tau \le -1$.

The power exponential distribution allows also a convenient expression for the Lorenz curve. Following GASTWIRTH (1971) we define it as

$$L(p) = \mu^{-1}\int_0^p F^{-1}(x)\mathrm{d}x \qquad 0 < p < 1$$

where $F^{-1}(t)$ denotes the quantile function and μ the finite mean. When applied to the power exponential distribution, we get:

$$L(p) = Q\big(1+\tau, -\ln(p)\big) \qquad -1 < \tau \le 0$$

$$= P\big(1+\tau, -\ln(1-p)\big) \qquad 0 \le \tau$$

$$= 1-(1-p)\sum_{i=0}^{\tau}\{-\ln(1-p)\}^i / i! \qquad \tau = 0,1,2,3,4,\cdots$$

Following DORFMAN (1979) the Gini index corresponding to this Lorenz curve boils down as:

$$G = 1-2\int_0^1 L(p)\mathrm{d}p = 1-\mu^{-1}\int_0^\infty \big(1-F(t)\big)^2\mathrm{d}t$$

$$= \mu^{-1}\int_0^\infty F(t)\big(1-F(t)\big)\mathrm{d}t = |1-2^{-\tau}| \qquad \tau > -1$$

Finally, following SHORROCKS (1980), we consider a generalised entropy as inequality measure that in case of the power exponential distribution results as:

$$\frac{E\left[\mu^{-c}T^c\right]-1}{c(c-1)} = \frac{\Gamma^{-c}(1+\tau)\cdot\Gamma(1+c\tau)-1}{c(c-1)} \qquad c \ne 0,1$$

$$= \frac{\tau\cdot\psi(1+c\tau)-\ln\Gamma(1+\tau)}{2c-1} \qquad c = 0,1 \quad \text{(Theil)}$$

where $\ln\Gamma$ denotes the logarithm of the Γ-function and ψ its first derivative, the digamma function.

4. MEAN VALUE OF ORDER STATISTICS

Let $T_{1:n}, T_{2:n}, \cdots, T_{n:n}$ be the order statistics of a random sample of size *n* from a Fréchet or Weibull distribution. The mean of such an order statistic is proportional with the population mean and can be displayed as

$$E(T_{r:n}) = n\mu\pi_{rn} \qquad r = 1,\cdots,n \qquad \tau > -1$$

where the factor shares π_{rn} form a monotone positive sequence. Rephrasing results of LIEBLEIN (1955) we may write:

$$\pi_{rn} = \binom{n-1}{r-1} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} (r+i)^{-1-\tau}$$

When $\tau \le 0$ we have $T_{1:n} < T_{2:n} < \cdots < T_{n:n}$ and $\pi_{1n} < \pi_{2n} < \cdots < \pi_{nn}$. When $\tau > 0$ these orderings are reversed. In both cases $\pi_{nn} = n^{-1-\tau}$ holds. For computational purposes this formula for π_{rn} becomes unstable as soon as $n > 20$. Embedding matters in matrix terms is a way out. We arrange the factor shares in an upper triangular matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} \\ & \pi_{22} & \cdots & \pi_{2n} \\ & & \ddots & \vdots \\ & & & \pi_{nn} \end{bmatrix}$$

We introduce the identity matrix $\mathbf{I}$, the diagonal matrices $\mathbf{J}$ and $\mathbf{D}$, a bidiagonal $\mathbf{B}$ and uppertriangular $\mathbf{C}$ and $\mathbf{U}$, all $n \times n$:

$$\mathbf{J} = \begin{bmatrix} -1 & & & \\ & (-1)^2 & & \\ & & \ddots & \\ & & & (-1)^n \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix}$$

$$\mathbf{C} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \qquad \mathbf{U} = (\mathbf{DB})^{-1} = \mathbf{CD}^{-1} = \begin{bmatrix} 1 & 2^{-1} & \cdots & n^{-1} \\ & 2^{-1} & & n^{-1} \\ & & \ddots & \vdots \\ & & & n^{-1} \end{bmatrix}$$

We will further need an uppertriangular matrix $\mathbf{P}$ containing Pascal's triangle:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ & 1 & 2 & & & n-1 \\ & & 1 & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & 1 & n-1 \\ & & & & & 1 \end{bmatrix} \qquad p_{ij} = \binom{j-1}{i-1} \qquad 1 \le i \le j \le n$$

$\mathbf{P}^{-1} = \mathbf{JPJ}$ see BELLMAN (1970) page 28.

Using paper and pencil combinatorics, we may rewrite Π as follows:

$$\Pi = \mathbf{P}^{-1}\mathbf{D}^{-1-\tau}\mathbf{P} = (\mathbf{P}^{-1}\mathbf{DP})^{-1-\tau} = (\mathbf{DB})^{-1-\tau} = \mathbf{U}^{1+\tau}$$

Using ι as a vector of ones, n times the last column of Π becomes $\mathbf{U}^{\tau}\iota$ and we may write:

$$\begin{aligned} \mu \cdot \mathbf{U}^{\tau}\iota &= E(\mathbf{t}^{\uparrow}) && -1 < \tau \le 0 \\ &= E(\mathbf{t}^{\downarrow}) && 0 \le \tau \end{aligned}$$

where $\mathbf{t} \in \Re^{n}$ denotes the original unordered power-exponential random sample and the suffix ↑ (↓) rearranges the components of a (random) vector in ascending (descending) order of magnitude. The problem is the evaluation of $\mathbf{U}^{\tau}\iota$. As **U** is triangular, the recursion as given in PARLETT (1976) suggests itself. This approach is also plagued however, due to the diagonal elements of **U** coming close to each other. Numerical stability is achieved by following the binomial expansion approach in WAUGH and ABEL (1967). The algorithm has the following ingredients:

$$\mathbf{A} = s\mathbf{U} - \mathbf{I} \qquad s = 2n/(n+1) \quad \text{minimizes spectral radius } \mathbf{A}$$

$$\mathbf{U} = s^{-1}(\mathbf{I} + \mathbf{A})$$

$$\mathbf{U}^{\tau} = s^{-\tau}\sum_{i=0}^{\infty}\binom{\tau}{i}\mathbf{A}^{i} = s^{-\tau}\left(\mathbf{I} + \tau\mathbf{A} + \frac{\tau(\tau-1)}{2!}\mathbf{A}^{2} + \frac{\tau(\tau-1)(\tau-2)}{3!}\mathbf{A}^{3} + \cdots\right)$$

$$\mathbf{w}_{i} = \binom{\tau}{i}\mathbf{A}^{i}\iota \qquad i = 0,1,2,3,\cdots$$

$$\mathbf{w}_{i+1} = \frac{\tau - i}{i+1}\mathbf{A}\mathbf{w}_{i} = \frac{i-\tau}{i+1}\left(\mathbf{w}_{i} - s\mathbf{CD}^{-1}\mathbf{w}_{i}\right) \quad i = 0,1,2,3,\cdots \qquad \mathbf{w}_{0} = \iota$$

$$\mathbf{U}^{\tau}\iota = s^{-\tau}\sum_{i=0}^{\infty}\mathbf{w}_{i}$$

Although the spectral radius of **A** is minimized, it still amounts to $(n-1)/(n+1)$ which approaches 1 for n large. This causes the need for a large number of terms in the evaluation of $\mathbf{U}^{\tau}\iota$. Luckily, the iterative evaluation of $\mathbf{w}_{i}$ only needs vector-vector operations, reversals and partial sums. So, it is matrix-free. Numerical stability prefers evaluation for $-1 < \tau \le 0$ and applying $\mathbf{U}^{1+\tau}\iota = \mathbf{C}\mathbf{D}^{-1}(\mathbf{U}^{\tau}\iota)$ as often as required.

5. Likelihood Analysis

For purposes of parameter estimation, paralleling the standard linear model, it is convenient to consider the logarithm of the power exponential variate *T*. The variate $Y = \ln T$ follows a Gumbel distribution with density given by

$$g(y \mid \tau, \theta) = \frac{1}{|\tau|}\exp\left[\frac{y - \ln\theta}{\tau} - \exp\left(\frac{y - \ln\theta}{\tau}\right)\right]$$

which for $\tau < 0\,(> 0)$ is an extreme value distribution for maxima (minima). We consider a sample of *n* observations from such a distribution. We allow the scale parameter θ to vary over the observations and model its logarithm as a linear model. We display the following matrices and vectors where [ ] denotes elementwise operation:

$$\ln[\theta] = \mathbf{X}\beta \qquad \mathbf{X} : n \times k \qquad \beta \in \Re^k$$
$$\mathbf{u} = \tau^{-1}(\mathbf{y} - \mathbf{X}\beta) \qquad \mathbf{w} = \exp[\mathbf{u}] \qquad \mathbf{w}_{\Delta} \text{ diagonal matrix such that } \mathbf{w}_{\Delta}\iota = \mathbf{w}$$
$$\mathbf{e}_1 = (1 \quad 0 \quad \cdots \quad 0)' \in \Re^{k+1}$$

Using the digamma and trigamma function, the mean and variance of **y** are given as

$$E(\mathbf{y}) = \mathbf{X}\beta + \tau\psi(1)\iota \qquad \psi(1) = -0.5772156649$$
$$V(\mathbf{y}) = \tau^2\psi'(1)\mathbf{I} \qquad \psi'(1) = \quad 1.6449340668 = \pi^2/6$$

which enables starting values for τ and β through least squares. We form *minus* the logarithm of the likelihood function

$$-\ln L(\tau, \beta) = n\ln|\tau| + \iota'(\mathbf{w} - \mathbf{u})$$

Writing $\mathbf{Q} = (\mathbf{u}, \mathbf{X})$ the gradient and Hessian can be displayed as

$$\mathbf{g} = \frac{\partial - \ln L(\tau, \beta)}{\partial\begin{bmatrix}\tau \\ \beta\end{bmatrix}} = \tau^{-1}\left(\mathbf{Q}'(\iota - \mathbf{w}) + n\mathbf{e}_1\right)$$

$$\mathbf{H} = \frac{\partial\mathbf{g}}{\partial(\tau, \beta')} = \tau^{-2}\left(\mathbf{Q}'\mathbf{w}_{\Delta}\mathbf{Q} + n\mathbf{e}_1\mathbf{e}_1'\right) - \tau^{-1}\left(\mathbf{g}\mathbf{e}_1' + \mathbf{e}_1\mathbf{g}'\right)$$

At a stationary point $\mathbf{g}=\mathbf{0}$ and $\mathbf{H}$ will be positive definite. The information matrix follows as

$$E(\mathbf{H})=\tau^{-2}\begin{bmatrix} n\pi^2/6+nc^2 & c\iota'\mathbf{X} \\ c\mathbf{X}'\iota & \mathbf{X}'\mathbf{X}\end{bmatrix} \qquad c=\psi(2)=1+\psi(1)=0.4227843351$$

The *computation* of maximum likelihood estimators might be slightly simplified by considering the following reparametrisation:

$$\begin{bmatrix}\tau \\ \beta\end{bmatrix}\to\gamma=\begin{bmatrix}\tau^{-1} \\ -\tau^{-1}\beta\end{bmatrix}$$

together with $\mathbf{Z}=(\mathbf{y},\mathbf{X})$ and $\mathbf{u}=\mathbf{Z}\gamma.$ This parametrisation is the same as the one used by OLSEN (1978) to prove uniqueness of the maximum likelihood estimator for the Tobit model. We form *minus* the logarithm of the likelihood function

$$-\ln L(\gamma)=\iota'(\mathbf{w}-\mathbf{u})-n\ln|\mathbf{e}_1'\gamma|$$

The gradient and Hessian for Newton-Raphson iteration are easily derived as:

$$\mathbf{g}=\frac{\partial-\ln L}{\partial\gamma}=\mathbf{Z}'(\mathbf{w}-\iota)-n\tau\mathbf{e}_1$$

$$\mathbf{H}=\frac{\partial\mathbf{g}}{\partial\gamma'}=\mathbf{Z}'\mathbf{w}_\Delta\mathbf{Z}+n\tau^2\mathbf{e}_1\mathbf{e}_1'$$

With this parametrisation, the Hessian will be positive definite everywhere in the parameter space. This parameter space splits in two halfspaces. One for the Fréchet-case $\Re_-\times\Re^k$ and one for the Weibull-case $\Re_+\times\Re^k$. The estimation criterion function is convex in γ in both halfspaces. So, if there are stationary points, these will correspond to unique maximum likelihood estimators. For the Weibull-case, the convexity property holds also when left-truncated or right-censored observations are present.

Taking for granted that the first column of $\mathbf{X}$ corresponds with a constant term, the starting value for the second element of γ can be improved as $\gamma_2\to\gamma_2-\ln(\iota'\mathbf{w}/n).$ Using these starting values and taking care that γ_1 does not change sign during the iteration steps, convergence will be swift and reliable. A further safeguard could be a check on improvement of the criterion function. In this way it is hard to imagine a situation where the numerical process will break down. Indeed, when the least squares starting values exist with a nonzero estimate for τ, things will be in order. For the simple case where $\mathbf{X}$ boils down to the constant term column, this applies for $n\geq 2$ with positive sample variance. The borderline n=2 has two stationary solutions for τ which are equal in absolute value.

6. FINAL REMARK

After the foregoing it seems fair to conjecture that other formulae using aspects of the Fréchet or Weibull distribution are liable for simplification too. Consider for instance the symmetric location-scale family put forward by BALAKRISHNAN and KOCHERLAKOTA (1985) and known as double Weibull. It is straightforward to include the double Fréchet and display its probability density as:

$$f(y\,|\,\mu,\sigma,\tau)=\frac{1}{2\sigma\,|\,\tau\,|}\left|\frac{y-\mu}{\sigma}\right|^{1/\tau-1}\exp\left[-\left|\frac{y-\mu}{\sigma}\right|^{1/\tau}\right]\qquad -\infty<y,\mu,\tau<\infty\quad \sigma>0$$

Its variance and kurtosis boil down as:

$$\text{variance}:\qquad \sigma^2\Gamma(1+2\tau)\qquad \tau>-\tfrac{1}{2}$$

$$\text{kurtosis}:\qquad \binom{4\tau}{2\tau}\geq 1\qquad \tau>-\tfrac{1}{4}$$

When $\tau<1$ this is a bimodal density with modes at $y=\mu\pm\sigma(1-\tau)^{\tau}$. This approaches a two-point distribution when $\tau\to 0$.

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