The Tipping Point: Low Rates and Financial Stability[a]

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Abstract

This paper studies the effect of low interest rates on financial stability. To this end, it develops a recursive model in which banks create liquidity. The model features two steady states: a good one with a healthy banking system and a bad one with a failed banking system. The paper's main result is the characterisation of a critical interest-rate level, below which a financial crisis takes place and the economy transitions from the good to the bad steady state. At this tipping point, the erosion of the net interest spread earned by banks outweighs the revaluation of banks' long-term assets, leading to insolvency. The value of the tipping point depends on two observable bank characteristics: the equilibrium net interest spread and the duration gap.

Keywords: Effective lower bound, franchise value of deposits, liquidity creation.

JEL Codes: E43, E50, G21.

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1 Introduction

Interest rates that are low by historical standard have prevailed in advanced economies since the Great Recession. Moreover, low interest rates are predicted to define the macroeconomic environment of the future (Kiley and Roberts, 2017). Thus, we should take a special interest in the effects of low rates on economic outcomes. This paper focuses on the effect of low rates on the stability of the financial system.

I base my study of financial stability on the canonical banking model developed in Diamond and Dybvig (1983). Banks are crisis-prone because they create liquidity. That is, banks issue more deposits than they hold assets. Hence, if all depositors try to withdraw at once, the bank fails. This paper focuses on crises that are driven by bad fundamentals, as opposed to self-fulfilling crises (Allen and Gale, 1998). In a fundamental crisis, banks fail to make good on their outstanding deposits even if a lender of last resort forestalls all self-fulfilling crises. Simply, the returns on bank assets are insufficient and banks are insolvent.

The canonical banking model is surprisingly well suited to study the effect of low rates on financial stability because of two elements that naturally emerge. First, the model has a lower bound on deposit rates. Banks do not offer a negative deposit rate, lest they trigger large withdrawals and thus unleash a financial crisis.[1] Second, banks earn a net interest spread in equilibrium between the interest rate on their assets and the deposit rate. These are the two key ingredients of the standard mechanism of low rates: low rates reduce the profitability of banks because banks cannot earn their targeted net interest spread once the zero lower bound (ZLB) on deposit rates is binding.[2] This mechanism is standard, in the sense that the literature agrees on it and starts from it to investigate the wider economic consequences of low rates on, for example, credit supply (Brunnermeier and Koby, 2018) and risk taking (Heider et al., 2019). My contribution is to embed this standard mechanism in a model in which banks are not merely credit intermediaries. Banks create liquidity and are therefore exposed to financial crises. In this framework, I can study the effect that low rates have on financial stability via the compression they exert on banks' net interest spreads.

In the model, subdued net interest spreads threaten financial stability by reducing banks' franchise value of deposits, i.e. the present discounted value of net interest

[1] In the context of the canonical banking model, the zero lower bound on deposit rates is usually referred to as the incentive-compatibility constraint. For the paper's results to hold, the lower bound does not have to be zero. Nonetheless, Heider et al. (2019) and Eggertsson et al. (2019) find strong evidence for a zero lower bound at least on the interest rates paid on retail deposits.

[2] There exists an empirical literature that documents this mechanism (e.g., Claessens et al., 2018). It is discussed in the subsection on related literature.

spreads over the lifetime of the average deposit. In fact, if a bank creates liquidity and thus has more deposits outstanding than assets, its solvency is founded on a high enough franchise value of deposits. Depositors count on the bank partly using retained profits to pay them off in the future when they withdraw their deposits. On the other hand, a reduction in the interest rate leads to an increase in asset values since banks hold long-term assets. Through this asset-revaluation effect, low rates tend to improve the balance-sheet position of banks and indeed bolster financial stability. If the former effect dominates the latter, depositors realise that the bank is insolvent. Thus, it is in their interest to run to the bank and withdraw all their deposits. This precipitates a financial crisis.

The main result of the paper is the characterisation of a cut-off level for the interest rate. If the interest rate falls below it, banks are tipped into insolvency and a financial crisis ensues. I call this cut-off interest rate the tipping point. At the tipping point, the adverse effect on the franchise value of deposits dominates the asset-revaluation effect and leads to bank insolvency. The tipping point depends on bank characteristics and has a simple analytical formula. It is given by banks' targeted net interest spread, a positive term, minus a term that captures the strength of the asset-revaluation effect. Hence, the tipping point is not necessarily positive nor necessarily negative. The strength of the asset-revaluation effect is increasing in the banks' effective duration gap at the ZLB.[3] In other words, it depends on how much more sensitive to a reduction in the interest rate the value of bank assets is relative to the value of deposits once the ZLB on deposit rates is binding.

Finding the empirical counterpart to banks' effective duration gap at the ZLB is the key to the paper's quantitative exercise, which estimates the tipping point. For the duration of bank assets, the empirical literature readily provides a value. On the other hand, the paper's analysis shows that the statutory duration of zero does not represent the effective duration of deposits. In fact, part of the value of deposits for a bank is their franchise value, which is increasing in the the interest rate. Using estimates in the literature to account for this, the paper calculates the effective duration of deposits at the ZLB to be 5.5 years. This is a remarkably large value. It follows that the asset-revaluation effect is relatively small. The estimate for the tipping point in the US economy in 2007 is 0.58%.[4]

[3]For an asset that consists of fixed payments, the duration is the weighted average of the times until the payments are received. Equivalently, it is the interest-rate elasticity of the value of the asset.

[4]This is an illustrative estimate of how low the US risk-free interest rate could have fallen permanently in September 2007 from its high point of 5.25% without destabilising banks. Notice that the tipping point is lower if the interest-rate reduction is not fully permanent.

Methodologically, the paper extends the canonical banking model along one dimension. It brings the time horizon to the infinite limit. Other papers adopted an overlapping-generations framework for this purpose (Bencivenga and Smith, 1991; Ennis and Keister, 2003). This paper is the first to consider infinitely-lived agents.[5] This approach has a key advantage: it makes the economy recursive. Therefore, I can analyse it with the tools of dynamic macroeconomics. It is possible to study the model's convergence properties and find that it has two steady states, one that is good in terms of welfare and one that is bad. For large enough shocks, the model generates dynamic paths that go from the good steady state to the bad. These are financial crises.

More generally, the recursive structure makes the model's endogenous objects stable over time. The tipping point is the same in every time period and the structure of bank balance sheets is too. This is both insightful and useful. A stable structure of bank balance sheets is especially insightful. We learn what liquidity creation is: banks have more deposits outstanding than assets. And we learn that a bank is solvent if its liquidity creation is backed up by a large enough franchise value of deposits. Stable endogenous objects are also useful because they can be matched with empirical counterparts. This makes quantification of the model possible.

Furthermore, an extended time horizon brings an advantage in terms of realism: it allows us to study financial crises when banks hold long-term assets, a defining characteristic of banking. In the canonical banking model with two periods, banks are crisis-prone only in the second one. Hence, when a financial crisis hits, bank assets have at most one period left before maturity. The extension of the time horizon carried out in this paper allows us to analyse the role of long-term assets in the event of a financial crisis. Hence, we can include the asset-revaluation effect in the analysis.

Related literature. A large empirical literature has studied and found evidence for the adverse effect of low rates on bank profits. Borio et al. (2017), Altavilla et al. (2018) and Claessens et al. (2018) look directly at banks' net interest margins and identify significant reductions due to low interest rates since the great financial crisis. Ampudia and van den Heuvel (2019) find that in the low-rate environment interest-rate cuts lead to reductions in bank valuations, particularly for banks more heavily funded with deposits. Much of the literature on low rates, including this paper, relies on this evidence to justify its analysis.

The wider implications of low rates have attracted a flurry of papers. Many

[5] Gertler and Kiyotaki (2015) and Segura and Suárez (2017) feature infinitely-lived agents but model liquidity risk differently from the canonical banking model. More discussion of this can be found under related literature.

find that financial institutions react to low interest rates by investing in riskier assets (Maddaloni and Peydró, 2011; Jiménez et al., 2014; Di Maggio and Kacperczyk, 2017; Martinez-Miera and Repullo, 2017; Basten and Mariathasan, 2018; Heider et al., 2019). Others find that in a low-rate environment banks reduce credit supply (Drechsler et al., 2017; Brunnermeier and Koby, 2018; Eggertsson et al., 2019; Amzallag et al., 2019). This paper abstracts from risk-taking and credit-supply considerations and contributes to the literature by focusing on liquidity creation and financial stability.

I apply a narrow definition of financial stability. In the footsteps of the seminal paper by Diamond and Dybvig (1983), I focus on financial fragility that is inherent in liquidity creation and study how this may give rise to crises. Within this paradigm, Gertler et al. (2016), Quadrini (2017) and Gertler et al. (2020) study the effects of panic-driven financial crises on macroeconomic outcomes. Fernández-Villaverde et al. (2020) focus on the effect of introducing a central-bank digital currency on both financial stability and output. In contrast, a large literature studies financial crises in the absence of liquidity creation. Examples of the latter literature are Bernanke et al. (1999), Gertler and Karadi (2011) and Mendicino et al. (2018).

The paper is related to a recent strand of literature that assesses the role of the franchise value of deposits in hedging banks' interest-rate risk. Di Tella and Kurlat (2017) and Drechsler et al. (2018) find that the statutory duration gap of bank assets over bank liabilities over-estimates the exposure of bank equity to changes in the interest rate.

Two papers in the literature make methodological contributions that are close to this paper's. Both Gertler and Kiyotaki (2015) and Segura and Suárez (2017) develop infinite-horizon models in which banks are exposed to costly financial crises. However, their modelling of liquidity risk departs from the canonical banking model. Respectively, in the former paper consumers are not subject to liquidity shocks and in the latter they are risk neutral. Moreover, illiquidity of bank assets is at the heart of the mechanism whereby financial crises are triggered in these models, as opposed to bank profitability in my paper. Therefore, while successful at capturing the effects of financial crises on economic outcomes, they are not suitable to study whether low rates, by reducing bank profitability, can trigger a financial crisis.

Paper outline. In the next section, I describe the environment and characterise the efficient allocation. In section 3, I set up the optimisation problems of the decentralised economy. Section 4 solves for the decentralised equilibrium and writes it in recursive form +. Section 5 characterises the steady states of the economy. The subsequent section interprets the balance sheets of banks engaging in liquidity creation and shows how the franchise value of deposits determines bank value. In section 7, I show that the economy can move from the good steady state to the bad one in response to a large enough change in the interest rate. I call this event a financial crisis and characterise the critical interest-rate level that triggers financial crises. The quantitative exercise is in section 8. Proofs of propositions, lemmas and corollaries are in the appendix.

2 Preferences and Technology

The economy is inhabited by a unit mass of infinitely-lived consumers who enjoy consumption according to utility function

$$\mathcal{U}\left(\{C_t\}_{t=0}^{+\infty}\right) = \lim_{T\to+\infty} \sum_{t=0}^{T} \prod_{j=0}^{t-1} (1-\theta_j)\cdot\theta_t\cdot u(C_t) + \prod_{j=0}^{T}(1-\theta_j)\cdot u\left(C_{T+1}\right). \tag{1}$$

The felicity function u features constant relative risk aversion $1/\alpha > 1$. The random variable θ_t represents an idiosyncratic liquidity shock, which takes on values 0 or 1. A consumer's probability of being hit by the liquidity shock is constant in every period, with $\Pr(\theta_t = 1) = \phi$. Notice that for $T = 0$ utility function (1) encompasses the utility function used in Diamond and Dybvig (1983) and most of the following literature. This is a generalisation to the infinite horizon.

Let me set some terminology. At every point in time, we call inactive those consumers who were hit by the liquidity shock in the past. As the name indicates, these consumers play no role in the dynamics of the model, because they value consumption neither in the current period nor in the future. The complementary group of consumers, who were not hit by the liquidity shock before, are called active. Active consumers can be further split into impatient and patient. At any point in time, a share ϕ of active consumers is hit by the liquidity shock. These consumers are impatient, in that they want to consume in the current period and are inactive from the next period on. The rest of the active consumers are called patient. They do not enjoy consumption yet but will at some point in the future. The quantity of active consumers N_t evolves deterministically over time according to $N_{t+1} = (1-\phi)\cdot N_t$.

There are two investment technologies: the productive technology and the storage technology. The productive technology is only available to banks. For any good invested in the productive technology, the bank chooses the time to maturity τ. The per-period net return is $\rho > 0$ regardless of the time to maturity of the investment. In other words, one good invested today in the productive technology with time to maturity τ yields

$(1+\rho)^\tau$ goods in τ periods. The other investment technology, available to both banks and consumers, is storage. A unit of output stored today gives a unit of output tomorrow. Notice that the productive technology is superior to storage. Banks can always invest for one period in the productive technology and earn the return $\rho > 0$.

Efficiency. Having described the preferences of consumers and the technologies available, we can determine the efficient allocation of consumption. First, two things to notice: (1) The productive technology is superior. Hence, the social planner does not store any goods. (2) Only consumers who are active and hit by the liquidity shock at a given date, i.e. impatient agents, enjoy consumption. Hence, the social planner only gives consumption C_t^i to impatient agents. Consumers who are either inactive or patient receive zero consumption.

At each point in time t, with the flow of resources $\tilde{K}_t(0)$ the social planner decides how much to invest in the productive technology at horizon τ, $\{I_t(\tau)\}_{\tau=1}^{+\infty}$, and how much consumption C_t^i to give to impatient consumers. The flow resource constraint is therefore

$$\sum_{\tau=1}^{+\infty} I_t(\tau) + \phi \cdot (1-\phi)^t \cdot C_t^i = \tilde{K}_t(0) \text{ for all } t \geq 0. \tag{2}$$

Investment at horizon τ, $I_t(\tau)$, increases the stock of resources that will become available at time $t+\tau$

$$\tilde{K}_{t+1}(\tau-1) = (1+\rho)^\tau \cdot I_t(\tau) + \tilde{K}_t(\tau) \text{ for all } t \geq 0 \text{ and } \tau \geq 1. \tag{3}$$

I define per-active-consumer wealth at time t as

$$K_t \equiv \frac{1}{(1-\phi)^t} \sum_{\tau=0}^{+\infty} \frac{\tilde{K}_t(\tau)}{(1+\rho)^\tau} \text{ for all } t \geq 0, \tag{4}$$

and impose non-negativity $K_{t+1} \geq 0$ for all $t \geq 0$.[6] Using the four conditions above and imposing a transversality condition

$$\lim_{t \to +\infty} \left(\frac{1-\phi}{1+\rho}\right)^t \cdot K_t \leq 0, \tag{5}$$

[6]It is useful to define per-active-consumer wealth. It is possible to reach the same results without this definition.

we can write the intertemporal resource constraint

$$\sum_{t=0}^{+\infty} \left(\frac{1-\phi}{1+\rho} \right)^t \cdot \phi \cdot C_t^i = K_0, \tag{6}$$

where K_0 is the initial level of wealth of the economy.[7]

Subject to the intertemporal resource constraint (6), the social planner chooses $\left\{ C_t^i \right\}_{t=0}^{+\infty}$ to maximise the sum of consumers' expected utility

$$\sum_{t=0}^{+\infty} \phi \cdot (1-\phi)^t \cdot u\left(C_t^i \right). \tag{7}$$

In the efficient allocation, the social planner smooths consumption across states for consumers. In particular, consumers are uncertain about the point in time when they will desire consumption. We call the uncertainty about time of consumption liquidity risk as is standard in the literature. The social planner could eliminate liquidity risk completely by giving each consumer the same level of consumption, regardless of their liquidity shock. But this behaviour reduces the size of the total pie. In fact, even with idiosyncratic liquidity risk, it is costly to provide liquidity-risk insurance, because giving consumption earlier in time implies forgoing future returns on investment. Hence, the optimal path of consumption features partial liquidity-risk insurance and implies

$$\frac{C_{t+1}^i}{C_t^i} = (1+\rho)^\alpha \quad t \geq 0, \tag{8}$$

where $\alpha \in (0, 1)$ is the inverse of the coefficient of relative risk aversion. Notice that full liquidity-risk insurance is a limiting parametric case. When the coefficient of relative risk aversion $1/\alpha \to +\infty$, then the social planner only cares about providing liquidity-risk insurance, regardless of the cost of doing so. On the other end of the spectrum, if the coefficient of relative risk aversion $1/\alpha \to 1$, then we have no liquidity-risk insurance. This liquidity-insurance motive is the same as in the canonical banking model. In fact, equation (8) is key for efficiency in that model, too.[8]

[7] The duration of wealth is irrelevant for welfare as long as the interest rate ρ is constant. The duration of wealth plays a key role in determining the economy's response to changes in the interest rate, as we will see in the next sections.

[8] Without loss of generality, I did not explicitly take into account consumers' private knowledge of their liquidity-shock realisation. Consumers have no incentive to mimic other types. Pretending to be an earlier consumer is sub-optimal, since later consumers get a weakly higher level of consumption by equation (8). Mimicking a later consumer is sub-optimal because consumers derive no utility from consuming after the time of their liquidity shock.

3 Decentralised Economy

Two agents inhabit the economy: consumers and banks. The key decisions are the consumers' deposit-withdrawal decision and the banks' deposit-rate setting.

3.1 Consumers

Consumers can save in bank deposits or in storage, not directly in the productive technology. The key decision that they make in every period is how much to withdraw from their deposit account. At any point in time, a share ϕ of active consumers are hit by the liquidity shock and become impatient. Their withdrawal decision is simple. Since they enjoy consumption today and become inactive in the following period, they want to consume all their wealth. Hence, they withdraw as much as they can: $W_t^i = D_t$. The more interesting withdrawal decision is made by the patient consumers, those active consumers not hit by the liquidity shock.

The withdrawal decision of a patient consumer is in truth a portfolio choice. By withdrawing they transform deposits into stored goods. The decision is based on the deposit rate and on the probability that the bank will fail to honour its deposits in the future. At a given time t, patient consumers choose withdrawals, next-period deposits and next-period storage $\left\{W_k^p, D_{k+1}, Z_{k+1}\right\}$ to maximise their expected utility

$$\mathbb{E}_t \sum_{k=t+1}^{+\infty} \phi \cdot (1-\phi)^{k-(t+1)} \cdot u\left(x_k \cdot D_k + Z_k\right), \tag{9}$$

where x_k represents a haircut that the bank applies to deposits if it fails at time k. A consumer's deposits evolve according to equation

$$D_{k+1} = (1+d_k) \cdot \left\{D_k - \left[x_k \cdot \mathbb{1}_{W_k^p \geq 0} + \left(1 - \mathbb{1}_{W_k^p \geq 0}\right)\right] \cdot W_k^p\right\}. \tag{10}$$

In each period, the consumer is offered a deposit rate on the deposit balance. She can change the deposit balance by withdrawing. If the bank does not fail in a given period k, then $x_k = 1$ and the consumer receives as many withdrawals as she demands. If the bank fails, the consumer receives only a share $x_k < 1$ of the withdrawals she demands. Of course, the haircut only applies to positive withdrawals. Stored goods evolve according to

$$Z_{k+1} = Z_k + \left[x_k \cdot \mathbb{1}_{W_k^p \geq 0} + \left(1 - \mathbb{1}_{W_k^p \geq 0}\right)\right] \cdot W_k^p. \tag{11}$$

If the patient consumer decides to have more stored goods, she can withdraw from the bank and keep the resources in storage. There is a limit to how much the consumer can withdraw from the bank. The consumer cannot withdraw more deposits than she has nor can she add to her deposit account more than she has in storage:

$$W_k^p \in [-Z_k, D_k]. \tag{12}$$

The initial levels of deposits D_t and storage S_t are given.

Optimal withdrawal decision. The patient depositor makes her deposit-withdrawal decision on the basis of whether storage or bank deposits offer a better return. For bank deposits, it is not only the deposit rate that she must consider. There are also her expectations about whether the bank will fail. The following lemma summarises the result.[9]

Lemma 1. *The optimal withdrawal decision of a patient consumer at a given time t is given by*

$$W_t^p(d_t) = \begin{cases} -Z_t & \text{if } \lambda_{D,t} \geq \lambda_{Z,t}, \\ D_t & \text{otherwise,} \end{cases} \tag{13}$$

where $\lambda_{D,t}$ is the value of a unit of deposits

$$\frac{\lambda_{D,t}}{1+d_t} = \mathbb{E}_t\left[\phi \cdot x_{t+1} \cdot u'(x_{t+1} \cdot D_{t+1} + S_{t+1}) + (1-\phi) \cdot \max\{\lambda_{D,t+1}, x_{t+1} \cdot \lambda_{Z,t+1}\}\right] \tag{14}$$

and $\lambda_{Z,t}$ is the value of a unit of stored goods

$$\lambda_{Z,t} = \mathbb{E}_t\left[\phi \cdot u'(x_{t+1} \cdot D_{t+1} + S_{t+1}) + (1-\phi) \cdot \max\{\lambda_{D,t+1}, \lambda_{Z,t+1}\}\right]. \tag{15}$$

A direct implication of the above lemma is that, if the deposit rate is set to a negative value, then patient consumers withdraw all of their deposits immediately.

Corollary 1. *At a given date t, if $d_t < 0$, then we have that $W_t^p(d_t) = D_t$.*

On the other hand, a non-negative deposit rate and the expectation that the bank will not fail in the next period are sufficient for patient consumers not to withdraw immediately their deposits.

Corollary 2. *At a given date t, if $d_t \geq 0$ and $\mathbb{E}_t(x_{t+1}) = 1$, then we have that $W_t^p(d_t) = -Z_t$.*

Expectations. Expectations play an important role in the withdrawal decision. In fact, as usual in this class of models expectations can be self-fulfilling. Consumers

[9] As a tie breaker, I assume that, if absolutely indifferent, the consumer decides not to withdraw.

who expect the bank to be unable to service its deposits in the next period withdraw everything today. Because the bank does not have enough resources at its disposal to pay back all of its deposits at once, these expectations cause the bank to fail today. This in turn verifies the expectations of the bank being bankrupt in the next period. The presence of sunspot equilibria of this kind has been studied in depth, starting from the seminal paper of the literature, Diamond and Dybvig (1983). Following Allen and Gale (1998), I do not focus on this type of equilibrium. Instead, I focus exclusively on crises that take place even under the most optimistic expectations. These are known in the literature as fundamental runs.

I formalise this notion by endowing consumers with expectations that, while model-consistent, are the most favourable to the banking system.

Assumption 1. *As long as it is model-consistent, at a given date t consumers hold expectations $\mathbb{E}_t(x_{t+k}) = 1$ for all $k \geq 1$.*

An interpretation of this assumption on expectations is that deposit insurance or a lender of last resort eliminate the purely panic-based runs on banks by coordinating expectations in the best possible way. The remaining runs are not due to coordination failures but to bad fundamentals.

3.2 Banks

Banks finance themselves offering demandable deposit contracts. This contract has two main characteristics: (1) deposits are non-contingent and (2) deposits are convertible on demand into goods. These characteristics are in conflict in case the bank does not have enough resources to service all withdrawals demanded. In this case, a bank-failure protocol is followed. The bank distributes all its resources to withdrawing depositors on a pro-rata basis.[10] Deposits that are demanded by consumers but not paid out by the bank remain as bank liabilities. The bank must pay them in the future with any assets it may come to have.

The bank's key decision is the deposit rate. At any point in time, it chooses the deposit rate to maximise consumers' welfare

$$\phi \cdot u(x_t \cdot D_t + Z_t) + (1-\phi) \cdot \mathbb{E}_t \sum_{k=t+1}^{+\infty} \phi \cdot (1-\phi)^{k-(t+1)} \cdot u(x_k \cdot D_k + Z_k). \tag{16}$$

[10]A first-come-first-served bank-failure protocol is also used in the literature. It has similar economic implications at the expense of a less tractable model, since it makes consumers heterogeneous in equilibrium after a financial crisis.

The bank internalises the impact of its deposit-rate choice on the per-active-consumer quantity of withdrawals W_k, given by

$$\mathrm{W}_k = \phi \cdot D_k + (1-\phi) \cdot W_k^p(d_k). \tag{17}$$

Therefore, banks are unwilling to set a negative deposit rate. The bank's assets evolve over time according to the law of motion

$$(1-\phi) \cdot \frac{B_{k+1}}{1+\rho} + x_k \cdot \mathrm{W}_k = B_k, \tag{18}$$

where B_k are bank's per-active-consumer assets at date k. Assets are partly used to service deposit withdrawals. The rest grows at the rate of return ρ and is adjusted to reflect the smaller active population in the following period. The variable x_k represents the share of deposit withdrawals that the bank actually pays out. When $x_k < 1$, we say that the bank failed. If total withdrawals are more than bank assets, then as according to the bank-failure protocol the bank pays out all of its assets to withdrawing depositors. Hence, we can write x_k as

$$x_k = \frac{\min\{\mathrm{W}_k, B_k\}}{\mathrm{W}_k}. \tag{19}$$

As a further constraint on the bank, we introduce an exogenous lower bound on the deposit rate:

$$d_k \geq -\phi. \tag{20}$$

Importantly, this exogenous lower bound on the deposit rate is not the zero lower bound on deposit rates, which arises endogenously. It is a strictly negative lower bound. Its role is exclusively to ensure that a failed bank cannot reduce its deposits with a large one-off reduction in the deposit rate.[11] The deposit-rate choice influences the evolution of bank deposits, in equation (10), and the evolution of stored goods, in equation (11).

The market for deposits is not perfectly competitive in the standard sense that at every point in time banks compete frictionlessly for deposits. The deposit contract is exclusive, in that consumers cannot defect to another bank at any point in time. At most, consumers can take their deposits out of their bank in order to store. As in the canonical banking model, there is perfect competition in the market for deposits in an ex-ante sense. At time 0, banks compete in offering a deposit contract to consumers. Then, the prevailing deposit contract maximises the consumers' expected utility. Hence, it offers the efficient level of liquidity-risk insurance and over time pays out all resources to

[11] This constraint limits the extent to which outstanding deposits can "jump" downwards.

consumers.[12]

In the next section, we solve for the bank's optimality conditions and characterise the economy's equilibrium.

4 Dynamics of the Model

The deposit contract stipulates that at each point in time the bank offer the deposit rate that maximises depositors' expected utility. In response to the deposit rate and to the creditworthiness of their bank, at each date consumers decide how much to withdraw from their deposit account. These are the key choices. In this section, I characterise the equilibrium decisions and show that the model's dynamics are recursive.

Lemma 2. *The equilibrium withdrawal decision at any point in time t is given by*

$$W_t^p = \begin{cases} -Z_t & \textit{if } \frac{B_t+Z_t}{D_t+Z_t} \geq \phi \cdot \frac{1+\rho}{\phi+\rho}, \\ D_t & \textit{otherwise.} \end{cases} \tag{21}$$

If $\frac{B_t+Z_t}{D_t+Z_t} \geq \phi \cdot \frac{1+\rho}{\phi+\rho}$, then the equilibrium deposit rate at time t is given by

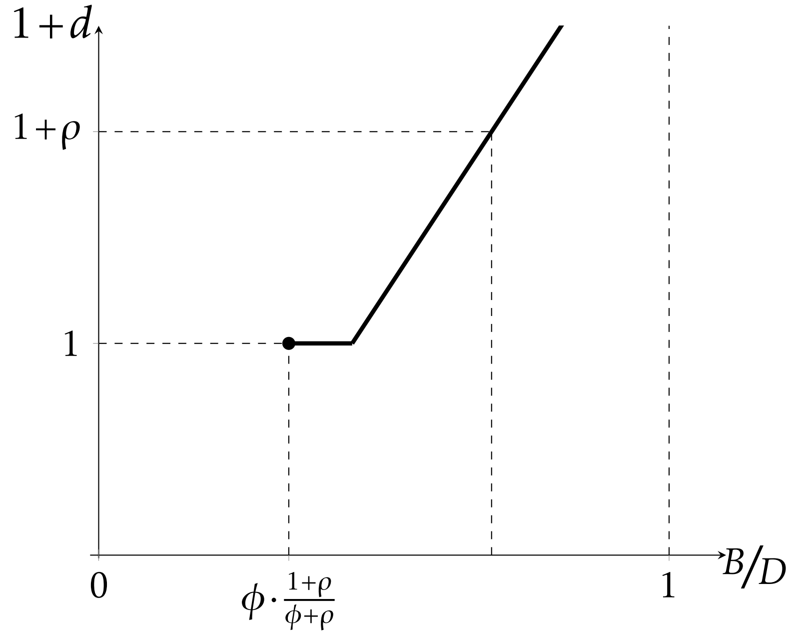
$$1 + d_t = \max\left\{(1+\rho)^{\alpha} \cdot \left(\frac{B_t+Z_t}{D_t+Z_t} - \phi\right) \cdot \frac{(1+\rho)^{1-\alpha} - (1-\phi)}{\phi \cdot (1-\phi)}, 1\right\}. \tag{22}$$

Otherwise, the deposit rate is undetermined in equilibrium.

The most noteworthy feature of the deposit-rate decision is the kink at zero. In principle, for some states of the world, for example for a sufficiently low interest rate ρ, the bank would like to set a negative deposit rate. However, it refrains from doing so, because consumers threaten to move their wealth into storage. Notice that the model generates this kink endogenously, because banks fear failure due to excessive withdrawals. We can visualise the deposit rate in figure 1 for the special case in which there are no goods in storage (i.e., $Z = 0$). The more assets the bank has relative to outstanding deposits, the higher the deposit rate it can afford to pay. For extremely low levels of bank assets, the deposit rate is undetermined because consumers withdraw everything, regardless of the deposit rate.

[12]As shown in Jacklin (1987), frictionless period-by-period competition in the market for deposits reduces welfare in this class of models, because the insurance mechanism becomes incentive incompatible. Farhi et al. (2009) show that, under the assumption that deposit contracts are exclusive, profit maximisation by banks at time 0 is equivalent to the maximisation of consumers' aggregate welfare.

Figure 1: Deposit rate with $Z = 0$.



It turns out that patient depositors withdraw according to a threshold strategy. If the bank has fewer assets per unit of deposit than a threshold, then the consumer can be sure that at some point in the future the bank will not be able to service demanded withdrawals. If a patient depositor expects that the bank will fail in the next time period, then it is in her best interest to anticipate the crisis and withdraw her deposits today. In equilibrium, all depositors withdraw all their deposits immediately. On the other hand, if the bank is above the threshold, the bank offers a non-negative deposit rate up until the infinite future and has enough resources to always honour withdrawals. Thus, it is in the consumers' interest to keep their wealth in the banking system. In this case, there is no financial crisis.

Recursive formulation. The economy has a recursive structure. The behaviour of banks and consumers does not directly depend on the time of their action but exclusively on the fundamentals of the economy. The variables that fully describe the state of the economy are bank assets B, banks' outstanding deposits D and the quantity of goods that consumers have in storage Z.

Lemma 3. *The bank's problem can be written recursively with value function*

$$V(B,D,Z) = \begin{cases} u(B+Z) & \text{if } \frac{B+Z}{D+Z} < \phi \cdot \frac{1+\rho}{\phi+\rho}, \\ \max\limits_{(d,B',D',Z')} \phi \cdot u(D+Z) + (1-\phi) \cdot V(B',D',Z') & \text{otherwise,} \end{cases} \tag{23}$$

subject to:

$$\frac{1-\phi}{1+\rho}\cdot B' = B - \phi\cdot D + (1-\phi)\cdot Z, \tag{24}$$

$$D' = (1+d)\cdot D, \tag{25}$$

$$Z' = 0, \tag{26}$$

$$d \geq 0. \tag{27}$$

The policy function is given by

$$1 + d(B,D,Z) = \max\left\{(1+\rho)^{\alpha}\cdot\left(\frac{B+Z}{D+Z}-\phi\right)\cdot\frac{(1+\rho)^{1-\alpha}-(1-\phi)}{\phi\cdot(1-\phi)}, 1\right\}. \tag{28}$$

Focusing on a state of the economy in which consumers do not hold goods in storage (i.e., $Z = 0$) helps to gain intuition into the problem. If banks hold too few assets relative to the deposits they owe, they are insolvent. In this case, consumers run on the bank regardless of the deposit rate on offer. They share the bank's assets on a pro-rata basis, move them into storage and consume them when they turn impatient. Above this threshold, banks are solvent. A negative deposit rate would trigger a run. Hence, as can be seen in the policy function, banks avoid it. When the zero lower bound is not binding, banks set the deposit rate to the level that implements optimal liquidity-risk insurance. Notice that, if consumers store goods and banks are not insolvent, then they find it optimal to deposit these goods in their bank account and at the following date there are no goods in storage.

Having proven that the economy is recursive, in what follows I drop the time indexation of endogenous variables unless clarity of exposition requires it.

5 Steady States

What state does the economy converge to over time? In this section, I show that the economy features three types of steady states and that, according to its initial state, it converges to one of them. A steady state is defined in the following way:

Definition 1. *A steady state is a state of the economy (B,D,Z) such that in the following trajectory state variables are either constant or grow at a constant rate.*

The following proposition outlines the characteristics of the different steady states.

Proposition 1. *The economy features three types of steady states:*

1. *The so-called **good steady states*** (B^*, D^*, Z^*) *have*

$$\frac{B^*}{D^*} = \frac{\phi \cdot (1+\rho)^{1-\alpha}}{(1+\rho)^{1-\alpha} - (1-\phi)}, \tag{29}$$

$$Z^* = 0. \tag{30}$$

2. *The so-called **unstable steady states*** $(\check{B}, \check{D}, \check{Z})$ *have*

$$\frac{\check{B}}{\check{D}} = \frac{\phi \cdot (1+\rho)}{\phi + \rho}, \tag{31}$$

$$\check{Z} = 0. \tag{32}$$

3. *The so-called **bad steady states*** $(B^\dagger, D^\dagger, Z^\dagger)$ *have*

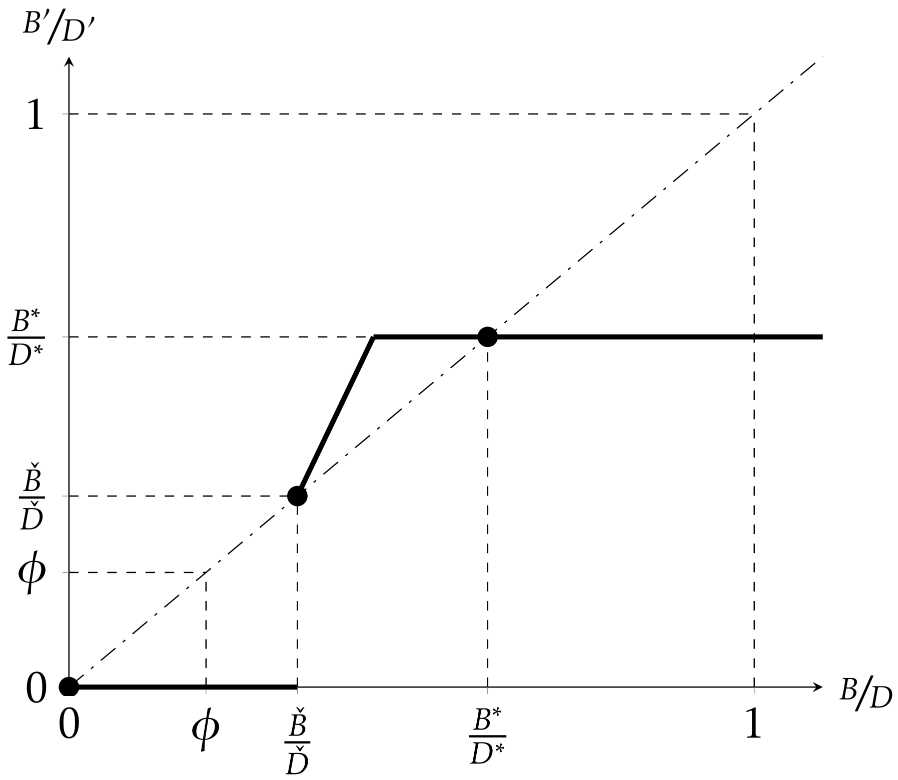
$$\frac{Z^\dagger}{D^\dagger} < \frac{\phi \cdot (1+\rho)}{(1-\phi) \cdot \rho}, \tag{33}$$

$$B^\dagger = 0. \tag{34}$$

Figure 2 helps us to conceptualise the existence of three types of steady states. If this ratio of bank assets to deposits is higher than in the good steady state, then the bank optimally sets a higher deposit rate in order to give more consumption to earlier consumers who have relatively high marginal utility of consumption. Thus, the economy converges to the good steady state. If the ratio is below, the bank reduces the deposit rate in order to reduce the consumption of earlier types. However, the bank faces a constraint in reducing the deposit rate. There is a lower bound at zero. Once this constraint is binding, convergence to the good steady state becomes slow. There is a point where the convergence to the good steady state takes an infinite amount of time. In other words, the bank can pay a zero deposit rate forever without accumulating or losing resources relative to its outstanding deposits. This is the unstable steady state. If the economy moves by an infinitesimal amount to the left of the unstable steady state, it would be certain that at some point in the future the bank would lack the resources to pay back its depositors in full. In this case, consumers optimally run immediately on the bank and the economy converges immediately to the bad steady state. In the bad steady state, consumers store their goods because the banking sector has failed.

Given an initial condition for the economy's resources $B + Z = Y$, we can rank the steady states in terms of welfare. The good steady state replicates the efficient

Figure 2: Law of motion for $\{B/D\}$ with $Z = 0$.



allocation that we analysed in section 2. Therefore, it is strictly better than the two other steady states. Both the unstable and bad steady state offer a flat inter-temporal profile of consumption to consumers. But, the unstable steady state offers more resources over time, since resources have better returns if held by banks rather than stored. The inter-temporal profile of consumption associated with the unstable steady state is therefore on a higher level and dominates the bad steady state's in terms of welfare.

As a function of the current state of the economy, we can identify what steady state the economy converges to over time. The following lemma lays out the result.

Lemma 4. *Consider a possible state of the economy* (B, D, Z).
If

$$\frac{B+Z}{D+Z} > \frac{\phi \cdot (1+\rho)}{\phi+\rho}, \tag{35}$$

then the trajectory converges to the good steady state.
If

$$\frac{B+Z}{D+Z} = \frac{\phi \cdot (1+\rho)}{\phi+\rho}, \tag{36}$$

then the trajectory converges to the unstable steady state.

Otherwise, the trajectory converges to the bad steady state.

Liquidity creation in steady state. On the good steady state, banks have more deposits outstanding than assets. As a consequence, if all consumers withdraw their deposits at once, then the bank would not have enough resources to satisfy them all. We call this behaviour liquidity creation.[13] Thanks to the existence of the banking sector, consumers have more assets in their hands. This sustains the consumption of earlier consumers, implementing liquidity-risk insurance. Thus, it is a socially desirable activity.

However, liquidity creation can only be sustained in equilibrium if deposits earn a lower interest than bank assets. This creates a limit to the amount of liquidity creation that can be performed and to the amount of liquidity creation that is socially desirable. Liquidity creation on the good steady state is the social optimum. On the unstable steady state, there is too much liquidity creation but it can still be sustained if deposits earn a zero deposit rate until the infinite future. More liquidity creation than on the unstable steady state cannot be sustained in equilibrium, since it gives depositors an incentive to withdraw everything from the bank.

Steady-state net interest spread. The net interest spread s is the difference between the interest rate that the bank receives on its assets and the interest rate that it pays to depositors:

$$1 + s \equiv \frac{1+\rho}{1+d}. \tag{37}$$

It is a measure of the bank's profitability and, as I will show in the next sections, is key in the determination of the level of interest rate at which banks can successfully operate.

It is important to notice that along the good steady state the net interest spread is strictly positive and in particular given by

$$1 + s^* = (1+\rho)^{1-\alpha}. \tag{38}$$

A positive net interest spread penalises consumers who hold their deposits for long more than it penalises consumers who withdraw relatively early. Thus, the banking system shares liquidity risk among its depositors. In this model, the socially optimal net interest spread is strictly positive.

[13] I use the name liquidity creation for this activity since it is already used in the literature. Perhaps, a more precise name for this activity could be deposit creation.

In the unstable steady state, the bank pays a zero deposit rate. Therefore, the net interest spread is larger than on the good steady state. It is given by $\check{s} = \rho$. In the bad steady state, banks have failed. Hence, the deposit rate is undetermined and so is the net interest spread.

6 Franchise Value of Deposits and Bank Equity

The economy that we described so far, in which banks have more deposits outstanding than assets, challenges our notion of balance sheets. The value of assets and liabilities must be matched, or else the bank's equity is negative. In what follows, I argue that this basic notion is of course valid. Yet, the banks in this model do not necessarily have negative equity. In fact, they only have negative equity along the bad steady state.

A missing asset accounts for this puzzle: the bank's deposit franchise. Over the lifetime of a deposit, the bank expects to earn a sequence of positive net interest spreads, defined in equation (37). This gives value to the bank and it reassures depositors that they will be paid back in full even if today the bank does not have enough assets to cover all of its outstanding liabilities. In other words, a bank performs liquidity creation and is solvent as long as the value of its deposit franchise is sufficiently high. If the franchise value of deposits falls, then indeed a bank's equity may turn negative and, as I show in this section, a financial crisis ensues.

I start by postulating an accounting identity

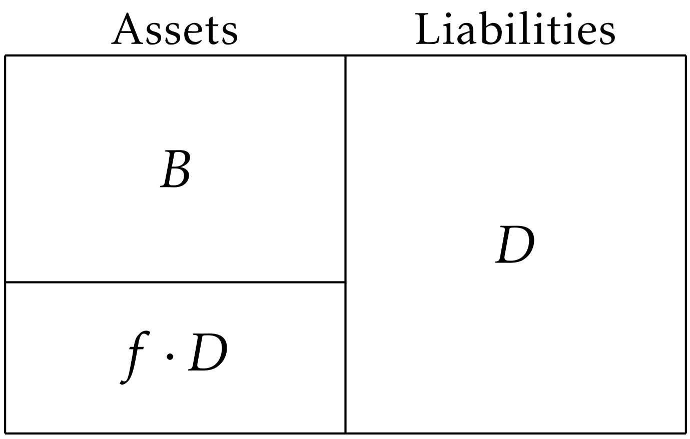
$$e_t \equiv B_t - (1 - f_t) \cdot D_t. \tag{39}$$

We know that B are bank assets and D is the face value of deposits outstanding. $f \cdot D$ is the value of the bank's deposit franchise. Banks fund themselves with deposits on which, in some states of the world and indeed in the good steady state, they pay a below-market interest rate. The capitalised value of these net interest spreads is the franchise value. The franchise value of a unit of deposits is defined as

$$f_t \equiv (1-\phi)\cdot\left(1-\frac{W_t^p}{D_t}\right)\cdot\left\{1-\sum_{k=t}^{+\infty}(1-\phi)^t\cdot\left[\phi+(1-\phi)\cdot\frac{W_{t+1}^p}{D_{t+1}}\right]\cdot\prod_{j=t}^{k}\left(1-\frac{W_j^p}{D_j}\right)\cdot\frac{1}{1+s_j}\right\}. \tag{40}$$

This is the summation of the net interest spreads enjoyed by the bank in the future weighted by the probability that each unit of deposits is not withdrawn from the bank earlier. With this in mind, we can interpret the formula as the average net interest

Figure 3: Bank balance sheet along the good steady state.



spread that the bank makes over the lifetime of a unit of deposits. I refer to $(1-f)\cdot D$ as the net cost of deposits for a bank. It is the face value of deposits minus the profits that the bank makes over the lifetime of those deposits.

Variable e is the residual that makes the accounting identity (39) hold. We can think of it as equity. That is, the value of the bank for a fictitious residual claimant. It is a meaningful concept to interpret the model, as shown by the following proposition.

Lemma 5. *If and only if the economy converges to the bad steady state, then we have that* $e_t < 0$.
Otherwise, we have that $e_t = 0$.

Interestingly, on all trajectories that converge to the unstable or to the good steady state, we have that the bank has exactly zero equity, $e = 0$. This result is tantamount to a zero-profit condition. All the bank's resources are eventually paid out to service withdrawals.

On the good steady state, the net interest spread is constant and consumers only withdraw their funds from the bank when they turn impatient. Hence, the franchise value of deposits is given by

$$f^* = \frac{1-\phi}{\phi + s^*} \cdot s^*. \tag{41}$$

The first factor is the average time to withdrawal of a deposit account in the bank. A higher net interest spread means that the deposit account does not accrue as much interest until it is withdrawn. Hence, the average time to withdrawal is decreasing in the net interest spread. The second factor is the net interest spread that the bank earns every period on deposits. The product of these two terms gives the franchise value of a unit of deposits. In a market in which banks trade deposits, this would be the highest price that a banks would be willing to pay for a deposit.[14]

7 Maturity Transformation and the Tipping Point

In this part of the analysis, I consider an unexpected and permanent change in the interest rate from the steady-state level ρ to $\hat{\rho}$. The key concept that emerges is the tipping point $\underline{\rho}$. If the interest rate moves below this cut-off level, the banking system fails and the economy moves from the good steady state to the bad steady state.

A low interest rate reduces the bank's franchise value of deposits. As discussed in the previous section, a large enough reduction pushes the banking system into insolvency. However, there is a force operating in the opposite direction. When banks hold long-term assets, a reduction in the interest rate results in an asset-revaluation effect. The tipping point is a result of the relative strength of these two effects.

Long-term assets have a positive duration. The textbook definition of duration is

$$\overline{\tau}_t \equiv \frac{1}{B_t} \cdot \sum_{\tau=0}^{+\infty} \tau \cdot \frac{\tilde{B}_t(\tau)}{(1+\rho)^{\tau}}, \tag{42}$$

where $\tilde{B}_t(\tau)$ are assets owned at time t maturing in τ periods and B_t the total value of bank assets at time t. Notice that I can treat the duration of bank assets as a parameter since it is not pinned down. As long as interest-rate shocks are not expected, the duration of bank assets is irrelevant from banks' perspective.

7.1 No maturity transformation

Let us start analysing the tipping point by assuming that the banking system only invests in short-term assets, i.e. $\overline{\tau} = 0$. Thus, we have no asset-revaluation effect. The only consequence of the unexpected reduction in the interest rate is a compression of the franchise value of deposits.

Of course, this is not a realistic representation of a bank, since maturity transformation is a defining function of banks. Nonetheless, it is useful to start from this case to better understand how the model works and the role of maturity transformation.

[14]In the unstable steady state, banks are solvent and consumers only withdraw when they become impatient. Banks are paying a zero deposit rate. In the bad steady state, all consumers withdraw everything that is left immediately. The franchise value of deposits in this case is therefore equal to zero.

Proposition 2. *Consider $\overline{\tau} = 0$. An economy starting in the good steady state converges to the bad steady state if and only if $\hat{\rho} < \underline{\rho}$ with*

$$\underline{\rho} = s^*. \tag{43}$$

The only determinant of the tipping point in this economy is the net interest spread that the banking system is earning in the good steady state. As long as the interest rate is greater than the net interest spread, the banking system can maintain its profitability in the face of a reduction in the rate of interest by correspondingly reducing the deposit rate. However, once the interest rate on bank assets is below the bank's targeted net interest spread, bank profitability starts falling, since it is impossible to cut the deposit rate further. This leads to a drop in bank equity and a financial crisis.

Notice that exactly at the tipping point, even if state variables are unchanged, the economy is no longer in the good steady state, since there is a new interest rate. It is in a tipping-point state, from which it starts converging to the new good steady state.

Corollary 3. *Once an economy on the good steady state with $\overline{\tau} = 0$ is hit by an interest-rate shock such that $\hat{\rho} = s^*$, the economy's equilibrium features $B = B^*$, $D = D^*$, $Z = 0$, $d = 0$, $f = f^*$ and $e = 0$. We call this the no-maturity-transformation tipping-point state.*

The above corollary fixes ideas about what happens to an economy without maturity transformation once it is at the tipping point. This helps us to add the effects of long-term assets to the analysis. In particular, we are interested in how the tipping point associated with the tipping-point state is related to the presence of long-term assets.

7.2 Effect of maturity transformation

If banks hold long-term assets, unexpected reductions in the rate of interest lead to an asset-revaluation effect. This pushes up bank equity and thus can protect the banking system from failure. In this subsection, we consider banks that hold assets with duration $\overline{\tau} \geq 0$.[15]

Bank equity is defined as the difference between the value of bank assets and the net cost of bank deposits:

$$e_t \equiv B_t - (1 - f_t) \cdot D_t. \tag{39}$$

[15] For simplicity, let us assume that the term structure of the bank assets in the good steady state has $\tilde{B}_t(\tau) = 0$ for all $\tau \neq \overline{\tau}$. This is an especially simple way for the bank to have an asset duration of $\overline{\tau}$. It means that the bank is only exposed to assets with a particular duration.

From lemma 5, we know that bank equity is key for the stability of banks. If the value of assets is insufficient to cover the net cost of bank deposits, then a financial crisis is inevitable. In what follows, we study how changes in the interest rate affect the value of assets and the net cost of deposits. By taking the difference of the two, we find the effect on bank equity and therefore on bank stability.

We focus on a first-order approximation around the no-maturity-transformation tipping-point state, defined in the previous subsection. The aim is to observe how an increase in bank-asset duration affects the endogenous variables when the economy is around this state. In particular, it is interesting to see to what extent it affects bank equity and the tipping point.

The value of bank assets can be approximated by

$$B \approx B^* + B^* \cdot \overline{\tau} \cdot \ln(1 + d^*), \tag{44}$$

where asset duration $\overline{\tau}$ is the interest-rate elasticity of asset value B and the steady-state deposit rate d^* represents the space for interest-rate reduction from the steady-state level ρ down to the tipping point. The product of the two gives the relative change in asset value once the interest rate has hit the tipping point.

The net cost of deposits can be approximated by

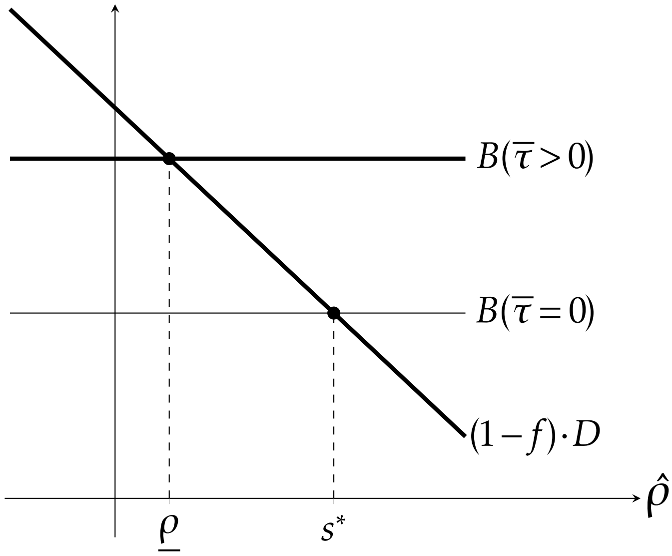
$$(1 - f) \cdot D \approx (1 - f^*) \cdot D^* - \frac{\partial f^*}{\partial s^*} \cdot D^* \cdot (\hat{\rho} - s^*), \tag{45}$$

where the partial derivative refers to equation (41). A change in the interest rate does not change the face value of deposits but the deposit franchise. In particular, once the economy is at the tipping point, the reduction in the interest rate on the franchise value of deposits translates directly in a reduction in the net interest spread. There can be no pass-through to the deposit rate, because the deposit rate is stuck at zero.

Figure 4 plots the value of bank assets and the net cost of bank deposits. The point at which the two schedules cross, where bank equity is zero, gives the tipping point. Any further reduction in the interest rate leads to negative equity and a financial crisis.

As expected from the results of the previous subsection, in the absence of maturity transformation the interest rate cannot fall below the bank's good-steady-state net interest spread level without causing a financial crisis. The more maturity transformation the bank performs the lower the tipping point. Intuitively, more maturity transformation leads to a greater revaluation effect. The greater the revaluation effect, the greater the resilience of banks to a squeeze to their net interest spread because their profits are

Figure 4: The tipping point with maturity transformation.



topped up by capital gains. Noticeably, the sign of the tipping point is not theoretically obvious. It depends on bank characteristics.

A more formal characterisation of the tipping point is given in the following proposition.

Proposition 3. *Consider $\overline{\tau}$ in a neighbourhood of zero. An economy starting on the good steady state converges to the bad steady state if and only if $\hat{\rho} < \underline{\rho}$, where*

$$\underline{\rho} = s^* - \Delta \cdot \ln(1 + d^*) \tag{46}$$

with

$$\Delta \equiv \frac{\overline{\tau}}{\frac{\partial f^* / \partial s^*}{1 - f^*}} > 0. \tag{47}$$

The proposition helps us to decompose the revaluation effect in two terms: the coefficient Δ and the good-steady-state deposit rate. The latter represents the extent to which the interest rate falls before the tipping point is hit.[16] The coefficient Δ turns out to be the appropriate concept of maturity transformation to calculate the tipping point: the bank's effective duration gap at the ZLB. To see this notice that the denominator is the interest-rate elasticity of the net cost of deposits for the bank, which is the duration of deposits.

[16]This is true to a first-order approximation. That is, it is true without accounting for the extra-space that the revaluation effect affords the bank.

The bank's effective duration gap at the ZLB is the ratio between asset duration and deposit duration. Hence, it is a duration *gap*. It is *effective* because the deposit duration is different from deposits' statutory duration of zero. Moreover, it is important to specify that it is the effective duration gap *at the ZLB* because the effective deposit duration at the ZLB is greater than the effective deposit duration in normal times. This is because at the ZLB there is no pass-through to deposit rates. A reduction in the interest rate reduces the net interest spread one-for-one. Hence, the derivative is not taken with respect to the interest rate but directly with respect to the net interest spread.

8 Quantitative Exercise

In this section, I estimate the tipping point on the eve of the interest-rate cuts that brought the Federal Reserve's policy rate from 5.25% in September 2007 to 0.25% in December 2008. Through the lenses of this model, how low could the interest rate have gone without destabilising banks?

There is only one term in the formula for the tipping point for which an empirical counterpart is not readily available. That is the effective duration of deposits at the ZLB. Due to the absence of an empirical estimate in the literature, we estimate it indirectly in the following way. Total differentiation of the definition of bank equity in equation (39) with respect to the interest rate gives us

$$-\frac{\mathrm{d}e/\mathrm{d}\rho}{e} = \frac{B}{e} \cdot \left(\bar{\tau} - \frac{\mathrm{d}f/\mathrm{d}\rho}{1-f} \right). \tag{48}$$

If bank liabilities had no duration, then the interest-rate elasticity of bank equity should be equal to the duration of bank assets times the leverage ratio. However, estimates of the interest-rate elasticity of bank equity and bank assets are inconsistent with zero-duration liabilities.[17] By means of event studies around the FOMC announcements between 1997 and 2007, English et al. (2018) estimate that the elasticity of bank-stock returns to an interest-rate level shock is −10.204. Moreover, they report that the repricing-maturity of bank assets, the best available proxy for bank-asset duration, in the period 1997-2007 was 4.46 years. Considering that the ratio of bank capital to total assets in the USA averaged 9.42% in the period 1998-2007, we infer an effective duration of bank deposits of 3.50 years. In normal times, a reduction in the interest rate by 25 basis points reduces a bank's franchise value of deposits to the extent that

[17] Drechsler et al. (2018) also notice this and attribute it to a high interest-rate elasticity of the net cost of deposits.

deposits are 0.875% more costly for the bank. [18,19]

Estimating the effective duration of deposits is not enough for the purposes of this section. We need the effective duration of deposits *at the ZLB*. This is larger than in normal times, because at the the zero lower bound on the deposit rate banks cannot pass the reduction in the interest rate to depositors at all. If we totally differentiate equations (37) and (41) and then re-arrange them, we can write that

$$\frac{\partial f^*/\partial s^*}{1-f^*} = \frac{\mathrm{d}f^*/\mathrm{d}\rho}{1-f^*} \cdot \frac{1}{1-(1+s^*)\cdot \partial d^*/\partial \rho}. \tag{49}$$

The effective duration of deposits at the ZLB is the effective duration in normal times adjusted for the absence of pass-through once the deposit rate is zero. Drechsler et al. (2018) produce an estimate of the change in deposit rates offered by US banks after a change in the fed funds rate. They find that the pass-through to deposit rates is 0.354. A 25 basis points increase in the fed funds rate leads the average bank to increase its deposit rate by 8.85 basis points.[20] Now, to calculate the effective duration of deposits at the ZLB, I only miss banks' net interest spread s^*. I take the M2 own rate, i.e. the weighted average of rates on checkable deposits, thrift saving deposits, money market mutual funds and small time deposits, as the economy's deposit rate.[21] Since this was 2.54% in September 2007, the net interest spread is 2.64%. It follows that the effective duration of deposits at the ZLB is 5.50 years. Once the deposit rate is zero, a reduction in the interest rate by 25 basis points increases the net cost of deposits by 1.37%.

With the empirical counterparts to all elements in the formula, we can quantify the tipping point. Its value is 0.58%. If we take this quantity seriously, it means that monetary authorities in the USA in September 2007 could permanently lower the risk-free interest rate from 5.25% to 0.58% without undermining the solvency of the banking system. Given the stylised nature of the model, which misses for example a capital buffer in the good steady state, I think this quantification should be thought of as illustrative rather than precise. Nonetheless, we should take seriously that, since the effective duration of bank liabilities is much larger than their statutory duration,

[18] Hutchison and Pennacchi (1996) estimate a structural model and find an effective duration for NOW accounts of 6.69 years and for MMDAs of 0.37 years.

[19] Banks could be hedging their interest-rate risk with derivatives. However, English et al. (2018) report that only a fraction of banks trade in interest-rate derivatives. And Begenau et al. (2015) document that banks use derivatives to the opposite effect: on average they increase their exposure to interest-rate risk.

[20] A branch of the literature emphasises that this pass-through is asymmetric (Driscoll and Judson, 2013; Yankov, 2014). Incorporating this would make the correction to the effective duration of deposits larger, since in normal times interest-rate cuts are associated with stronger pass-through.

[21] The M2 own rate is published by the Federal Reserve Board on a monthly basis.

banks do not benefit much from asset-revaluation effects associated with interest-rate reductions. Using the correct concept of duration for deposits matters hugely in a quantitative sense. English et al. (2018) report an average repricing-maturity of bank liabilities in the period 1997-2007 of 0.41 years. We can think of this as the statutory duration of deposits. It implies a tipping point of -25%.

9 Conclusion

Bank profits are important for macroeconomic outcomes because they allow banks to supply more credit (Bernanke et al., 1999). This is an extremely influential idea, which shapes the discussion about the consequences of weak bank profitability on economic outcomes. This paper shows that healthy bank profits are key to macroeconomic performance for another reason too: because banks create liquidity. Liquidity creation means that at any point in time banks do not have enough assets to redeem all of their outstanding deposits. For this purpose, they partly rely on future profits. If bank profitability falls and it becomes clear that deposits do not have sufficient backing, a financial crisis is inevitable.

Suppose banks need to make at least a 1% net interest spread every period to back up their deposits. Then, as long as the interest rate remains above 1%, they are surely solvent. They can adjust the deposit rate to achieve their target net interest spread. Once the interest rate dips below 1%, their net interest spread is compressed but they can survive thanks to capital gains on their long-term assets. However, as the interest rate falls further down, the compression in the net interest spread outweighs the revaluation effect on long-term assets. The interest-rate level at which this happens is the tipping point. At this point, a low interest rate generates a financial crisis. The characterisation of the tipping point in terms of bank characteristics is the main result of the paper. This is followed by a discussion of the appropriate empirical counterparts to them and a quantification.

Methodologically, this paper extends the canonical banking model to an infinite horizon in a way that makes the economy recursive. On its own terms, this represents a contribution to the literature. First, it makes the mechanism more transparent by making the equilibrium objects, for example the bank's balance sheet and bank profits, stable over time. Second, objects that are stable over time are suitable for comparison with empirical counterparts and for quantification. Therefore, the recursive banking model, as presented in this paper, is in my view a useful tool for future research on financial stability.

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A Proofs

Proof of lemma 1. Taking $\{x_s, d_s\}_{s=t}^{+\infty}$ as given, the patient consumer chooses variables $\left\{W_s^p, Z_{s+1}, D_{s+1}\right\}_{s=t}^{+\infty}$ to maximise (9) subject to constraints (10), (11) and (12). The initial conditions D_t and Z_t are given. The Kuhn-Tucker conditions of this problem confirm the lemma.

Proof of corollary 1. Notice that $x_s \leq 1$ for all $s \geq t$, since the bank never pays more than the demanded withdrawals. It follows directly that, if $d_t < 0$, then $\lambda_D < \lambda_Z$.

Proof of corollary 2. Follows directly from plugging $d_t \geq 0$ and $\mathbb{E}_t(x_{t+1})$ in the expression for deposit withdrawals given in lemma 1.

Proof of lemma 2. First, guess that it is optimal for the bank to set $d_s \geq 0$ for all $s \geq t$. We verify this guess in the proof.

Given depositors' beliefs, the bank's problem can be written recursively. In particular, let's write two problems.

Problem 1. If depositors are optimistic, that is $\tilde{E}_t(x_s) = 1$ for all $s \geq t+1$, then at each point in time $W_t^p = -Z_t$. The problem is represented by

$$V(B,D,Z) = \max_{(d,B',D',Z')} \phi \cdot u(x \cdot D + Z) + (1-\phi) \cdot \tilde{E}[V(B',D',Z')] \tag{50}$$

subject to:

$$\frac{1-\phi}{1+\rho} \cdot B' = B - x \cdot [\phi \cdot D - (1-\phi) \cdot Z], \tag{51}$$

$$D' = (1+d) \cdot (D+Z), \tag{52}$$

$$Z' = 0, \tag{53}$$

$$x = \frac{\min\{\phi \cdot D - (1-\phi) \cdot Z, B\}}{\phi \cdot D - (1-\phi) \cdot Z}, \tag{54}$$

$$B' \geq 0. \tag{55}$$

Problem 2. If depositors are pessimistic, that is $\tilde{E}_t(x_s) = 0$ for all $s \geq t+1$, then at each point in time $W_t^p = D_t$. The problem is represented by

$$V(B,D,Z) = \max_{(d,B',D',Z')} \phi \cdot u(x \cdot D + Z) + (1-\phi) \cdot \tilde{E}[V(B',D',Z')] \tag{56}$$

subject to:

$$\frac{1-\phi}{1+\rho} \cdot B' = B - x \cdot D, \tag{57}$$

$$D' = (1+d) \cdot (1-x) \cdot D, \tag{58}$$

$$Z' = Z + x \cdot D, \tag{59}$$

$$x = \frac{\min\{D, B\}}{D}, \tag{60}$$

$$B' \geq 0. \tag{61}$$

In every given region of the state space, we must find the model-consistent beliefs. Remember that by assumption, as long as they are model-consistent, consumers hold optimistic beliefs.

Consider $\frac{B_t+Z_t}{D_t+Z_t} \leq \phi$. Guess that depositors hold optimistic beliefs. Hence, we focus on problem 1. By equations (51) and (54), clearly $x_{t+1} = 0$. Thus, in this region of the state space, optimistic beliefs are not model-consistent.
Guess that the depositors hold pessimistic beliefs. Hence, we focus on problem 2. By equations (57) and (60), clearly $x_{t+1} = 0$. For any admissible deposit rate $d \geq -\phi$, the economy remains in this region of the state space in the next period, since

$$\frac{B'+Z'}{D'+Z'} = \frac{B+Z}{D+Z+d\cdot(D-B)} \leq \phi. \tag{62}$$

Since it is not possible for the economy to emerge from this region, $x_s = 0$ for all $s \geq t+1$ and pessimistic beliefs are verified as model-consistent. It follows that the optimal d is indeterminate and the value function is $V(B,D,Z) = u(B+Z)$.

Consider $\phi < \frac{B_t+Z_t}{D_t+Z_t} < \phi \cdot \frac{1+\rho}{\phi+\rho}$. Guess that depositors hold optimistic beliefs. Hence, we focus on problem 1. Using the problem's constraints we can trace out

$$\frac{B_s+Z_s}{D_s+Z_s} = \phi + \left(\frac{1+\rho}{1-\phi}\right)^{s-t} \cdot \frac{1}{\prod_{j=t}^{s-1}\left(1+d_j\right)} \cdot \left(\frac{B_t+Z_t}{D_t+Z_t} - \phi\right) + \\ -\phi\cdot(1-\phi)\cdot\sum_{j=t}^{s-1}\left(\frac{1+\rho}{1-\phi}\right)^{j-t} \cdot \frac{1}{\prod_{k=s-(j-t)}^{s-1}(1+d_k)}. \tag{63}$$

Consider $d_s = 0$ for all $s \geq t+1$. At the finite time

$$\hat{s} = -\frac{\ln\left[1 - \frac{\phi+\rho}{\phi\cdot(1-\phi)} \cdot \left(\frac{B_t+Z_t}{D_t+Z_t} - \phi\right)\right]}{\ln\left(\frac{1+\rho}{1-\phi}\right)}, \tag{64}$$

the economy enters the region $\frac{B+Z}{D+Z} \leq \phi$ analysed above. For higher deposit rates, the economy reaches the region even earlier. It follows that optimistic beliefs are not model-consistent.

Guess that the depositors hold pessimistic beliefs. Hence, we focus on problem 2. By equations (57) and (60), clearly $x_{t+1} = 0$. For any admissible deposit rate $d \geq -\phi$, the economy remains in this region of the state space, since

$$\frac{B'+Z'}{D'+Z'} = \frac{B+Z}{D+Z+d\cdot(D-B)} < \phi \cdot \frac{1+\rho}{\phi+\rho}. \tag{65}$$

Since it is not possible for the economy to emerge from this region, $x_s = 0$ for all $s \geq t+1$ and pessimistic beliefs are verified as model-consistent. It follows that the optimal d is indeterminate and the value function is $V(B,D,Z) = u(B+Z)$.

Consider

$$\frac{B_t+Z_t}{D_t+Z_t} \geq \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha}\cdot[(1+\rho)^{1-\alpha} - (1-\phi)]}\right\}. \tag{66}$$

Guess that depositors hold optimistic beliefs. Hence, we focus on problem 1. Under these beliefs we can re-write the problem as

$$V(B,D,Z) = \max_{(d,B',D',Z')} \phi \cdot u(D+Z) + (1-\phi)\cdot \tilde{E}\left[V(B',D',Z')\right] \tag{67}$$

subject to:

$$D'+Z' = (1+d)\cdot(D+Z), \tag{68}$$

$$\frac{1-\phi}{1+\rho}\cdot B' = B+Z-\phi\cdot(D+Z), \tag{69}$$

$$\lim_{s\to+\infty}\left(\frac{1-\phi}{1+\rho}\right)^s \cdot B_s = 0. \tag{70}$$

The optimal deposit rate is given by

$$1+d = (1+\rho)^{\alpha}\cdot\left(\frac{B+Z}{D+Z} - \phi\right)\cdot\frac{(1+\rho)^{1-\alpha} - (1-\phi)}{\phi\cdot(1-\phi)}. \tag{71}$$

By substitution, we can verify that the optimal deposit rate in this region is non-negative. We can verify that the optimistic beliefs are model-consistent using equation (63). We find that

$$\frac{B_s + Z_s}{D_s + Z_s} = \frac{\phi \cdot (1+\rho)^{1-\alpha}}{(1+\rho)^{1-\alpha} - (1-\phi)} > \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha} \cdot [(1+\rho)^{1-\alpha} - (1-\phi)]}\right\} > \phi. \tag{72}$$

Hence, once the economy is in this region of the state space, it stays in it. Moreover, $x_s = 1$ for all $s \geq t+1$. The value function is then

$$\begin{aligned} V(B,D,Z) = {} & \phi \cdot u(D+Z) + \\ & + \phi \cdot \sum_{s=1}^{+\infty} (1-\phi)^s \cdot u\left[(1+\rho)^{\alpha \cdot s} \cdot \left(\frac{B+Z}{D+Z} - \phi\right) \cdot \frac{(1+\rho)^{1-\alpha} - (1-\phi)}{\phi \cdot (1-\phi)} \cdot (D+Z)\right]. \end{aligned} \tag{73}$$

Consider

$$\phi \cdot \frac{1+\rho}{\phi+\rho} \leq \frac{B_t + Z_t}{D_t + Z_t} < \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha} \cdot [(1+\rho)^{1-\alpha} - (1-\phi)]}\right\}. \tag{74}$$

Guess that depositors hold optimistic beliefs. Suppose $d_s = 0$ in this region, from equation (63) we can verify that at a finite time

$$\hat{s} = \frac{\ln\left\{\frac{(1-\phi)\cdot[(1+\rho)^{\alpha}-1]}{(1+\rho)^{\alpha}\cdot\left[(1+\rho)^{1-\alpha}-(1-\phi)\right]\cdot\left[\frac{B_t+Z_t}{D_t+Z_t}\cdot\frac{\phi+\rho}{\phi\cdot(1-\phi)}-1\right]}\right\}}{\ln\left(\frac{1+\rho}{1-\phi}\right)} \tag{75}$$

the economy reaches the region with $\frac{B_t+Z_t}{D_t+Z_t} \geq \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha}\cdot\left[(1+\rho)^{1-\alpha}-(1-\phi)\right]}\right\}$ analysed above. Indeed, in this region $d = 0$ is optimal, as the economy emerges fastest into the region where the deposit rate is not constrained by zero. Setting the deposit rate greater than zero is suboptimal, because it worsens liquidity-risk insurance. Setting the deposit rate negative is suboptimal, because by corollary 1 it leads patient depositors to withdraw everything and makes the economy converge to the region with $\frac{B+Z}{D+Z} \leq \phi$ discussed above.

Proof of lemma 3. Refer to the proof of lemma 2. Notice that in the case with $\frac{B+Z}{D+Z} < \phi \cdot \frac{1+\rho}{\phi+\rho}$ we have that $V(B,D,Z) = u(B+Z)$. In the rest of the state space, problem 1 applies, since optimistic expectations are model-consistent, and the deposit rate is given by equation (22). The lemma follows from this.

Proof of proposition 1. Let us start from state space characterised by $\frac{B+Z}{D+Z} < \phi \cdot \frac{1+\rho}{\phi+\rho}$. From lemma 2, we know that there is a bank run immediately and therefore

$$B_j = 0 \quad \text{for all } j \geq t+1. \tag{76}$$

Hence, a constant growth rate of the state variables requires an initial condition $B^{\dagger} = 0$. For the condition on the state space to hold, the initial conditions must also comply with

$$\frac{Z^{\dagger}}{D^{\dagger}} \leq \frac{\phi \cdot (1+\rho)}{(1-\phi) \cdot \rho}. \tag{77}$$

These are the bad steady states.

Now, consider the state space characterised by $\frac{B+Z}{D+Z} \geq \phi \cdot \frac{1+\rho}{\phi+\rho}$. First of all, since $Z' = 0$, a constant growth rate implies an initial condition $Z^{ss} = 0$. Using equation (24) we can see that B_j^{ss} and D_j^{ss} that satisfy a steady state must comply with

$$1 + g^B = \frac{1+\rho}{1-\phi} \cdot \left(1 - \phi \cdot \frac{D_j^{ss}}{B_j^{ss}}\right) \quad \text{for all } j \geq t. \tag{78}$$

The expression implies that a steady state requires a constant B_j^{ss}/D_j^{ss} for all $j \geq t$. This implies that B and D grow at the same rate in a steady state. With this in mind, we can write that the initial levels B^{ss} and S^{ss} that satisfy a steady state must also comply with

$$1 + g^B = \max\left\{(1+\rho)^{\alpha} \cdot \left(\frac{B_j^{ss}}{D_j^{ss}} - \phi\right) \cdot \frac{(1+\rho)^{1-\alpha} - (1-\phi)}{\phi \cdot (1-\phi)}, 1\right\}. \tag{79}$$

It is easy to check that the system of non-linear equations above has only two solutions that correspond to the good and to the unstable steady state.

Proof of lemma 4. Let us start from the state space characterised by $\frac{B+Z}{D+Z} < \phi \cdot \frac{1+\rho}{\phi+\rho}$. According to lemma 2, there is an immediate bank run. This implies $B' = 0$, $Z' = (1-\phi) \cdot (B+Z)$ and $D' = (1+d) \cdot (D-B)$. Since

$$\frac{Z'}{D'} = \frac{1-\phi}{1+d} \cdot \frac{B+Z}{D+Z} \cdot \frac{1}{1 - \frac{B+Z}{D+Z}} < \frac{\phi \cdot (1+\rho)}{(1-\phi) \cdot \rho}, \tag{80}$$

we have that in this state space the economy moves to a bad steady state in one period.

Consider the state space characterised by $\frac{B+Z}{D+Z} \geq \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha} \cdot \left[(1+\rho)^{1-\alpha} - (1-\phi)\right]}\right\}$. Substituting it in the policy function for the deposit rate (22), we have that the economy moves to the good steady state in one period.

Consider

$$\phi \cdot \frac{1+\rho}{\phi+\rho} < \frac{B+Z}{D+Z} < \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha} \cdot [(1+\rho)^{1-\alpha} - (1-\phi)]}\right\}. \tag{81}$$

From equation (63) we can verify that at a finite time

$$\hat{s} = \frac{\ln\left\{\frac{(1-\phi)\cdot[(1+\rho)^{\alpha}-1]}{(1+\rho)^{\alpha}\cdot\left[(1+\rho)^{1-\alpha}-(1-\phi)\right]\cdot\left[\frac{B_t+Z_t}{D_t+Z_t}\cdot\frac{\phi+\rho}{\phi\cdot(1-\phi)}-1\right]}\right\}}{\ln\left(\frac{1+\rho}{1-\phi}\right)} \tag{82}$$

the economy reaches the region with $\frac{B_t+Z_t}{D_t+Z_t} \geq \phi \cdot \left\{1 + \frac{1-\phi}{(1+\rho)^{\alpha} \cdot \left[(1+\rho)^{1-\alpha} - (1-\phi)\right]}\right\}$ analysed above. Then, it moves to a good steady state in one period.

Ultimately, consider

$$\phi \cdot \frac{1+\rho}{\phi+\rho} = \frac{B+Z}{D+Z}. \tag{83}$$

In this case, the economy is in an unstable steady state.

Proof of lemma 5. Remember that e is defined by

$$B + f \cdot D \equiv e + D. \tag{39}$$

By lemma 4, we know that the economy converges to the bad steady state if

$$\frac{B+Z}{D+Z} < \phi \cdot \frac{1+\rho}{\phi+\rho}. \tag{84}$$

From the definition of the franchise value of deposits (40), we have that $f = 0$. It follows that $e < 0$.

In the rest of the state space, in which the economy converges either to the unstable or the good steady state, substituting the policy function (22) in the definition of the franchise value of deposits (40), we have that $f = \frac{D-B}{D}$. It follows that $e = 0$.

Proof of proposition 2. If the economy starts on the good steady state, using the result given in lemma 4 we find that, after the interest-rate changes from ρ to $\hat{\rho}$, the economy converges to the bad steady state if and only if

$$\frac{\phi \cdot (1+\rho)^{1-\alpha}}{(1+\rho)^{1-\alpha} - (1-\phi)} < \phi \cdot \frac{1+\hat{\rho}}{\phi+\hat{\rho}}. \tag{85}$$

This is true for any $\hat{\rho} < \underline{\rho}$ with $1 + \underline{\rho} = (1+\rho)^{1-\alpha}$. Notice that the good-steady-state net interest spread is given by $1 + s^* = (1+\rho)^{1-\alpha}$.

Proof of corollary 3. The state variables (B, D, Z) do not change because they are independent of the interest rate. Notice that asset duration is zero. Choice variable d is found by plugging the state variables and the tipping-point interest rate, which from proposition 2 we know is $\underline{\rho} = s^*$, in the policy function (28).

Proof of proposition 3. Using the definition of bank-asset duration (42), we can write the bank asset value as

$$\frac{B}{B^*} = \left(\frac{1+\rho}{1+\hat{\rho}}\right)^{\overline{\tau}}. \tag{86}$$

The first-order Taylor expansion around the no-maturity-transformation tipping-point state corresponds to equation (44).

A first-order Taylor expansion of the cost of deposits around the no-maturity-transformation tipping-point state is given by

$$(1-f) \cdot D \approx (1-f^*) \cdot D^* + \left(\left.\frac{\partial f}{\partial \rho}\right|_{\hat{\rho}=s^*, \overline{\tau}=0} + \left.\frac{\partial f}{\partial d}\right|_{\hat{\rho}=s^*, \overline{\tau}=0} \cdot \left.\frac{\partial d}{\partial \rho}\right|_{\hat{\rho}=s^*, \overline{\tau}=0} \right) \cdot (\hat{\rho} - s^*). \tag{87}$$

Notice that in the no-maturity-transformation tipping-point state $d = 0$. A further reduction in the interest rate has no effect on the deposit rate. The left-hand derivative of the deposit rate with respect to the interest rate is equal to zero. Since we look at an interest rate below s^*, we can write that $\partial d/\partial \rho \,\big|_{\hat{\rho}=s^*, \overline{\tau}=0} = 0$. Using equation (41), the expression can be written as (45).

At the tipping point, according to lemma 5 bank equity e is equal to zero. With this constraint and equations (44) and (45), we have that the only possible tipping point is given by equation (46).