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Systematic risk under extremely adverse market conditions

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Abstract Extreme losses are the major concern in risk management. The dependence between financial assets and the market portfolio changes under extremely adverse market conditions. We develop a measure of systematic tail risk, the *tail regression beta*, defined by an asset's sensitivity to large negative market shocks, and establish the estimation methodology. We compare it to regular systematic risk measures: the market beta and the downside beta. Furthermore, the tail regression beta is a useful instrument in both portfolio risk management and systemic risk management. We demonstrate its applications in analyzing Value-at-Risk (VaR) and Conditional Value-at-Risk (CoVaR).

Keywords: Tail regression beta, downside risk, Extreme Value Theory, tail dependence, risk management.

JEL Classification Numbers: C14, G11

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1 Introduction

Risk managers and prudential regulators often assess portfolio risk by the portfolio's sensitivity to key risk factors. The market beta is commonly given the most prominent position in this assessment. As a measure of the sensitivity to systematic risk, the market beta provides the proportional change of the asset return to the change of market factor. Nevertheless, both practitioners and academics have realized the limitations of this approach for risk management purposes. One critique is that the market beta assesses the sensitivity to the market irrespective of market conditions. By contrast, risk managers are concerned with extreme losses in distress events, the so-called *tail events*. Building on extreme value theory, we provide a methodology to estimate the sensitivity to risk factors during extremely adverse market conditions as the *tail regression beta* and further address the weakness of the regular market beta in risk management. The scope of this paper comprises the estimation of the tail regression beta and its applications in risk management. We apply the methodology in an empirical exercise on industry portfolios.

The changing dependence structure in adverse market conditions may be a potential reason on why the portfolio's sensitivity to the market risk, market beta, depends on market conditions. It is a well-known stylized fact that equity returns demonstrate higher correlations during periods of high stock market volatility, see e.g. King and Wadhwani (1990), Longin and Solnik (1995), Karolyi and Stulz (1996) and Ramchand and Susmel (1998). In addition, correlations increase especially during periods of severe market downturns, as reported by Longin and Solnik (2001), Ang and Bekaert (2002), Ang and Chen (2002) and Patton (2004). The observation of increased correlations in bear markets is a major concern in risk management, as increased correlations dissolve the benefits of portfolio diversification.

A first attempt to assess the systematic risk under different market conditions is to focus on market downturns. Based on the equilibrium model of Bawa and Lindenberg (1977), Price et al. (1982) estimate the *downside beta*, which is the market beta conditioned on below average market returns. The downside beta of assets is found to be significantly different from their regular market beta. Following this result, several studies apply the concept of downside beta to explain the cross-section of returns, such as Harlow and Rao (1989), Ang et al. (2006), Pedersen and Hwang (2007), Estrada (2007) and Galagedera (2007).

Although focusing on downside beta is a first step into the downside direction, the estimation is still based on a large number of observations from common trading days. As aforementioned, for risk management purposes it is necessary to assess the dependence in severe market conditions. Several studies step in with asymptotic dependence measures based on Extreme Value Theory (EVT), see Poon et al. (2004), Hartmann et al. (2004) and de Jonghe (2010) among others. The asymptotic dependence measures play a similar role in the tail as the correlation coefficient does at a moderate level. Nevertheless, they do not provide the sensitivity to market risk under extremely adverse market conditions as market beta does regardless of market conditions.

To distinguish a dependence measure from a sensitivity measure, consider the following differences between correlation and market beta. First, correlation indicates the potential comovements obtained from the joint distribution, while market beta captures the co-movement at the absolute level, i.e. if the market moves one percent, then the asset is expected to move 'beta' percent. Second, when constructing a portfolio consisting of multiple assets, it is difficult to aggregate the correlations between the individual assets and the market factor into a correlation between the portfolio and the market factor. However, with the market beta, a portfolio beta is simply the weighted average of the asset betas in the portfolio. Third, correlation is by definition a measure independent from marginal risks, whereas the market beta is a measure combining correlation with marginal risks. The characteristics of correlation can be shifted to the asymptotic dependence measures when analyzing tail However, a sensitivity measure on the systematic risk under extremely adverse events. market condition is still lacking. We fill the gap by developing a measure that is the tail equivalent to market beta. We will show that this measure, the tail regression beta, has the similar interpretation and additivity properties as the market beta and takes into account the marginal tail risk of assets. For sake of clarity, we distinguish the market beta irrespective of market conditions as the regular beta. The differences between correlation and regular beta can be shifted to differences between asymptotic dependence measures and the tail regression beta.

In our estimation methodology we exploit the heavy tail feature of financial returns instead of applying a linear regression based on tail observations. Simulations show that our estimation methodology yields an estimator that has a lower mean squared error than performing regressions in the tail. Theoretically, we find that the estimator of the tail regression beta has a similar structure as the estimator of the regular beta from regression analysis. The estimator of the regular beta consists of a dependence measure, given by the correlation, and the marginal risk measures, given by the standard deviations. In the estimator of the tail regression beta, the dependence measure is replaced by an asymptotic dependence measure and the marginal risk measures are replaced by the Value-at-Risk (VaR) that is obtained from tail observations. The empirical results demonstrate that the regular portfolio sensitivity to the systematic risk is in general different from the sensitivity to systematic risk in severe market downturns.

The setup of the paper is as follows. Section 2 describes the estimation methodology and reports several simulation results. Section 3 provides an empirical application of the tail regression beta on industry portfolios. In this section the tail regression beta is shown to be considerably different from the regular beta for several industry portfolios. Remarkably, these differences are not captured by differences between the downside beta and the regular beta. Section 4 discusses applications of the tail regression beta in risk management. We show how to use the tail regression beta to estimate the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CoVaR) of a portfolio. Section 5 concludes.

2 Methodology

2.1 Estimating the tail regression beta

We consider the regular single factor model on asset returns as

$$R_t^e = \beta R_t^m + \varepsilon_t,$$

where R_t^e denotes the excess return of an asset, R_t^m denotes the excess market return, and ε_t is a well-behaved error term. The regular risk analysis assesses the potential (co)variation of the asset return and the market return regardless of market conditions. By regarding R_t^m as the market risk and ε_t as the idiosyncratic risk of the asset, the coefficient β measures the sensitivity of the asset to the market variation. It is regarded as a systematic risk measure of the asset regardless of the market condition. With historical data, the regular beta can be estimated from regression analysis.

When considering the downside risk only, the systematic risk measure can be different from that of the regular regression analysis. Bawa and Lindenberg (1977) propose the downside beta defined in the model

$$R_t^e = \beta^{DS} R_t^m + \varepsilon_t, \quad \text{for } R_t^m < 0.$$

Roughly speaking, the single factor model is considered only in case that the market is in a downturn. The coefficient β^{DS} measures how much the asset is exposed to the downside risk of the market. It can be regarded as the systematic risk measure of the asset *vis-à-vis* the downside risk of the market. A regression which takes only observations with negative excess market returns into the analysis can produce a proper estimate for the coefficient β^{DS} .

By further lowering the threshold in defining "downside", we consider a tail regression which focuses on the linear relation between the market return and the asset return in their downside tails. More specifically, the downside linear relation is given as in the following model,

$$R_t^e = \beta^T R_t^m + \varepsilon_t, \quad \text{for } R_t^m < -u, \tag{2.1}$$

where u is a high threshold such that the tail probability that the market return falls below -u is at a low level. Parallelled to the aforementioned two cases, the coefficient in the tail regression, β^T , measures the sensitivity of the asset return to the tail risk of the market portfolio. It is thus regarded as the systematic risk measure under extremely adverse market condition.

In order to make statistical inference on the coefficient β^T , it is necessary to model the downside tail of the distributions of both the market risk and idiosyncratic risk. We use heavy-tailed distributions for that purpose. The heavy-tailedness of tail distributions of financial returns is well-documented in literature, see e.g. Jansen and De Vries (1991), Embrechts et al. (1997), etc. In correspondence with the heavy-tailed feature, we assume that as $u \to \infty$

$$\Pr(R^m < -u) \sim A_m u^{-\alpha_m} \text{ and } \Pr(\varepsilon < -u) \sim A_\varepsilon u^{-\alpha_\varepsilon}.$$
 (2.2)

Here, the parameters α_m and α_{ε} are the tail indices, while the parameters A_m and A_{ε} are the scales. As ε represents the idiosyncratic risk, it is independent from the market risk R^m . Suppose the model (2.1) holds for a larger area, $\min(R^m, \varepsilon) < -u$, from the Feller theorem (see Feller (1971)), we have that as $u \to \infty$

$$\Pr(R^e < -u) \sim \Pr(\beta^T R^m < -u) + \Pr(\varepsilon < -u)$$

Together with the heavy-tail feature in (2.2), we have that in the two distinguished cases $\alpha_m < \alpha_{\varepsilon}$ and $\alpha_m > \alpha_{\varepsilon}$, the two terms on the right hand side dominate the downside tail of R^e respectively, while in the case $\alpha_m = \alpha_{\varepsilon}$, both of them contribute to the tail distribution. To summarize, we obtain the heavy-tailedness of the downside tail distribution of the excess

asset return as follows. As $u \to \infty$,

$$\Pr(R^e < -u) \sim Au^{-\alpha},\tag{2.3}$$

where $\alpha = \min(\alpha_m, \alpha_{\varepsilon})$, and

$$A = \begin{cases} (\beta^T)^{\alpha_m} A_m, & \text{if } \alpha_m < \alpha_{\varepsilon}, \\ A_{\varepsilon}, & \text{if } \alpha_m > \alpha_{\varepsilon}, \\ (\beta^T)^{\alpha_m} A_m + A_{\varepsilon}, & \text{if } \alpha_m = \alpha_{\varepsilon} = \alpha. \end{cases}$$
(2.4)

The heavy-tail feature of the downside return is convenient for marginal tail risk analysis because the tail index and the scale parameter can be estimated from historical observations. Thus, we can estimate the parameters α_m , A_m , α and A. However, estimating marginal information is not sufficient to estimate the coefficient β^T in the tail regression. To see this, consider the two possible cases: $\alpha_m \leq \alpha_{\varepsilon}$ and $\alpha_m > \alpha_{\varepsilon}$. On the one hand, if $\alpha_m \leq \alpha_{\varepsilon}$, empirical analysis would not reject the null that $\alpha = \alpha_m$. From estimating α and α_m , we can not distinguish whether it is the first or the third scenario in (2.4). Hence, the coefficient β^T is unidentified. On the other hand, if $\alpha_m > \alpha_{\varepsilon}$, empirical analysis would tend to reject the null of $\alpha = \alpha_m$ and subsequently identify that it is in the second scenario in (2.4). However, the estimation of marginal scales only provides information on the scale of the idiosyncratic risk which does not help estimate β^T either. To summarize, in order to estimate β^T , it is necessary to have more information than the marginal tail distributions.

Asymptotic dependence measures can provide the additional information that is needed to make statistical inference on β^T . This is similar to the role of estimating the correlation coefficient in regular regression analysis. More specifically, the linear model in (2.1) imposes dependence structure on the asset return and the market return in the tail: if the market suffers a severe loss, then it is likely to observe a severe loss in the individual asset. Such an phenomenon is described as tail dependence. We consider an asymptotic dependence measure which stems from the multivariate EVT, as follows

$$\tau := \lim_{p \to 0} \tau(p) := \lim_{p \to 0} \frac{1}{p} \Pr(R^e < -VaR_e(p) \text{ and } R^m < -VaR_m(p)),$$

where $VaR_e(p)$ and $VaR_m(p)$ are the VaRs of the asset return and the market return at probability level p. For any given low tail probability p, the VaR is defined by $Pr(R^e < -VaR_e(p)) = p$. In other words, having a large loss that is higher than VaR(p) is a tail event with probability p. The τ measure has a clear economic interpretation towards contagion risk: by rewriting it as

$$\tau = \lim_{p \to 0} \Pr(R^e < -VaR_e(p) | R^m < -VaR_m(p)),$$

the τ measure indicates the probability of having a large loss on the asset conditional on an extremely adverse market situation. Since τ is the limit of a conditional probability, we have that $0 \leq \tau \leq 1$. The case $\tau = 0$ is regarded as tail independence, while the case $\tau = 1$ corresponds to completely tail dependence.¹ Moreover, we remark that τ is a measure of tail dependence regardless of the individual tail risk. Thus, it contains no information on the marginal distributions of R^e and R^m . The aforementioned features of the τ measure indicate that it plays a similar role as the correlation coefficient at a moderate level. The difference is that τ only measures the dependence in the tails. Multivariate EVT provides estimates on the τ measure when $\tau > 0$, see e.g. de Haan and Ferreira (2006). The τ measure is applied to different financial datasets in order to measure tail dependence, see e.g. Straetmans et al. (2008) and de Jonghe (2010).²

¹The τ measure is closely related to the measure $E(\kappa|\kappa \ge 1)$ introduced by Embrechts et al. (2000) and applied in Hartmann et al. (2004). Here κ is the number of tail events which is defined as having a large loss that corresponds to tail probability p. Thus, the measure $E(\kappa|\kappa \ge 1)$ is the expected number of tail events given that there is at least one which is also an alternative measure on tail dependence. It is not difficult to verify that, considering (R^m, R^e) , the two measures are connected by $E(\kappa|\kappa \ge 1) = \frac{2}{2-\tau}$. We prefer the τ measure since the measure $E(\kappa|\kappa \ge 1)$ treats the two dimensions symmetrically, while the τ measure can be interpreted as a conditional probability.

²In these two studies, the τ measure was named "tail beta". However, as we demonstrate, it is an analogous measure of the correlation coefficient rather than the regular beta in the regression analysis. On

Under the linear model in (2.1), it is easy to obtain that if $\alpha_m \leq \alpha_{\varepsilon}$, R^m and R^e are tail dependent, i.e. $\tau > 0$. The following proposition shows how to calculate the τ measure in this case. The proof is in Appendix A.

Proposition 2.1 Under the single factor model in (2.1) and the heavy-tail setup of the downside distributions (2.2), when $\alpha_{\varepsilon} \geq \alpha_m$ and $\beta^T \geq 0$, we have that $\alpha = \alpha_m$ and

$$\tau = \lim_{p \to 0} \left(\beta^T \frac{VaR_m(p)}{VaR_e(p)} \right)^{\alpha_m}.$$
(2.5)

With further assuming the heavy-tailed setup on the excess asset return in $(2.3)^3$, we have that

$$\tau = \frac{(\beta^T)^{\alpha_m} A_m}{A}$$

To estimate β^T , we first discuss the estimation of the two parameters α_m and τ . For the tail index, a widely-applied estimator is the so-called *Hill estimator* proposed in Hill (1975) as follows. With independent and identical distributed observations $X_1^{(m)} = -R_1^m, \dots, X_n^{(m)} =$ $-R_n^m$, by ranking them as $X_{n,1}^{(m)} \leq X_{n,2}^{(m)} \leq \dots \leq X_{n,n}^{(m)}$, the Hill estimator is defined as

$$\frac{1}{\hat{\alpha}_m} := \frac{1}{k} \sum_{i=1}^k \log X_{n,n-i+1}^{(m)} - \log X_{n,n-k}^{(m)}, \tag{2.6}$$

where k := k(n) is an intermediate sequence such that as $n \to \infty$, $k \to \infty$ and $k/n \to 0$. For the τ parameter, multivariate EVT provides a nonparametric estimate by a counting measure, which is defined as

$$\hat{\tau} := \frac{1}{k} \sum_{i=1}^{n} \mathbb{1}_{\left\{X_i > X_{n,n-k} \text{ and } X_i^{(m)} > X_{n,n-k}^{(m)}\right\}},$$
(2.7)

the contrary, the tail regression coefficient β^T proposed in this paper plays an analogous role to the regular beta but focuses only on the tail exposure. Thus, we compare the tail regression coefficient to the regular beta instead of considering the τ measure.

³Assumption (2.3) is necessary to obtain further calculation with respect to the scales. However, to prove (2.5), the tail regression model (2.1) is sufficient, see Appendix A.

where $X_{n,n-k}$ is the k + 1-th highest order statistic of the observations $X_1 := -R_1^e, X_2 := -R_2^e, \cdots, X_n := -R_n^e$.

To apply formula (2.5), we take the k + 1-th highest order statistics as the estimator of the VaR at a tail probability level $k/n \to 0$ as $n \to \infty$. By inverting equation (2.5), β^T is then estimated as

$$\hat{\beta}^T := \hat{\tau}^{1/\hat{\alpha}_m} \frac{X_{n,n-k}}{X_{n,n-k}^{(m)}}.$$
(2.8)

For the estimates on the marginal tail index, the parameter τ and the marginal VaRs, the usual statistical properties such as consistency and asymptotic normality hold, see de Haan and Ferreira (2006). Thus a direct consequence is that the estimator of β^T in (2.8) is consistent and asymptotic normal distributed.

Following the linear model in (2.1), if $\alpha_m > \alpha_{\varepsilon}$, we have that the tail risk of the asset is dominated by the idiosyncratic risk. This implies that $\tau = 0$. The case $\tau = 0$ is usually referred to as tail independence. With $\tau = 0$, the probability of a joint tail event is of a higher order than that of an individual tail event which is characterized by the probability p. However, it can still be modeled as in a series of papers by Ledford and Tawn (1996, 1997, 1998, 2003). Within the current notation, the Ledford and Tawn model leads to the assumption that for some $0 < \eta < 1$,

$$\lim_{p \to 0} p^{-1/\eta} \Pr(R^e < -VaR_e(p), R^m < -VaR_m(p)) \text{ exists and is positive.}^4$$

Compared to the definition of $\tau(p)$, such an assumption implies that as $p \to 0$, $p^{1-1/\eta}\tau(p)$ has a positive finite limit. Since the $\tau > 0$ case corresponds to $\eta = 1$, the Ledford and Tawn model with $0 < \eta < 1$ is a natural extension to the case $\tau = 0$. It is easy to obtain that $\eta = 1/2$, if R^e and R^m are completely independent. Moreover, in the case $1/2 < \eta < 1$ there

⁴The Ledford and Tawn model was extensively studied as a natural model of dependence under tail independent case. It has been discussed under the name "hidden regularly variation" in Resnick (2002); Maulik and Resnick (2004); Heffernan and Resnick (2005). A full characterization of such a model is given in de Haan and Zhou (2010). Draisma et al. (2004) studied the estimation of the η parameter. For application, Poon et al. (2004) applied both tail dependence and the Ledford and Tawn model in modeling financial returns from major stock indices.

is still potential "positive" co-movements of the tail events which can not be neglected when modeling dependence in extreme losses.⁵ Hence, now focus on the case $1/2 < \eta < 1$.

The following proposition shows the relation between $\tau(p)$ and the marginals under the model in (2.1) in the case $\alpha_{\varepsilon} < \alpha_m < 2\alpha_{\varepsilon}$ and establishes the link to the Ledford and Tawn model with $1/2 < \eta < 1$. The proof is again in Appendix.

Proposition 2.2 Under the single factor model (2.1) and the heavy-tail setup of the downside distributions (2.2), when $\alpha_{\varepsilon} < \alpha_m < 2\alpha_{\varepsilon}$ and $\beta^T \ge 0$, we have that as $p \to 0$,

$$\tau(p) \sim \left(\beta^T \frac{VaR_m(p)}{VaR_e(p)}\right)^{\alpha_m}.$$
(2.9)

With further assuming the heavy-tailed setup on the excess asset return in (2.3), we have that

$$\lim_{p \to 0} p^{1 - \alpha_m / \alpha_{\varepsilon}} \tau(p) = \lim_{p \to 0} p^{1 - \alpha_m / \alpha_{\varepsilon}} \left(\beta^T \frac{VaR_m(p)}{VaR_e(p)} \right)^{\alpha_m} = \frac{(\beta^T)^{\alpha_m} A_m}{A^{\alpha_m / \alpha_{\varepsilon}}}$$

In other words, it corresponds to the Ledford and Tawn model with $\eta = \alpha_{\varepsilon}/\alpha_m$.

Proposition 2.2 indicates that although $\tau(p)$ converges to zero as $p \to 0$ in the case $\alpha_{\varepsilon} < \alpha_m < 2\alpha_{\varepsilon}$, we can still follow (2.9) and use the estimator in (2.8). Combining the case $\alpha_m \leq \alpha_{\varepsilon}$ and the case $\alpha_{\varepsilon} < \alpha_m < 2\alpha_{\varepsilon}$, the estimator in (2.8) is valid regardless the tail dependency between the asset return and the market return. We state the property of the estimator in the following theorem. The proof is in Appendix A.

Theorem 2.3 Suppose the single factor model in (2.1) and the heavy-tail setup of the downside distributions (2.2) hold with $\alpha_m < 2\alpha_{\varepsilon}$.⁶ For a suitable intermediate sequence k = k(n), define the estimator of β^T as in (2.8) in which we use the estimator $\hat{\tau}$ as in (2.7) and the Hill estimator $\hat{\alpha}_m$ as in (2.6). Then it is a consistent estimator of β^T specified in the model.

We remark that the structure of the estimator $\hat{\beta}^T$ in (2.8) is comparable with the estimator

⁵It can be verified that if $1/2 < \eta < 1$, then $\tau(p)/p \to \infty$ as $p \to 0$.

 $^{^{6}}$ We remark again that assuming the heavy-tail feature of the excess asset return, (2.3), is not necessary for this theorem.

of beta in the regular regression analysis. When estimating the regular beta by regression analysis, the ordinary least square (OLS) estimator is given as

$$\hat{\beta} = \hat{\rho} \frac{\hat{\sigma}_e}{\hat{\sigma}_m},$$

where $\hat{\rho}$ is the estimated correlation coefficient between R^e and R^m , $\hat{\sigma}_e$ and $\hat{\sigma}_m$ are the estimated standard deviations of R^e and R^m respectively. The estimator $\hat{\beta}$ consists of three components, the dependence measure, and two measures on the individual risks of the asset return and market return. Similarly, the estimator $\hat{\beta}^T$ consists of the tail dependence measure $\hat{\tau}$, and two tail risk measures – the VaRs at probability level k/n – of the asset return and market return. The only additional element in the estimator $\hat{\beta}^T$ is the tail index $\hat{\alpha}_m$. By comparing the structure of the estimators, we observe that the estimator of the tail regression beta does. However, the tail regression beta focuses on the sensitivity to market risk in extremely adverse market conditions.

2.2 Simulations

We run simulations to examine the performance of the estimation procedure on the tail regression coefficient β^T . An alternative approach to tackle the same problem as the tail regression is to consider OLS regression analysis based on the observations corresponding to a large market loss only. More specifically, it is regarded as a threshold regression based only on those observations where the loss on the market return does exceed the threshold $VaR_m(p)$. The threshold regression provides the *threshold beta* which has been referred to as the second-order co-lower partial moment (CLPM) for a low probability threshold, see e.g. Post and Versijp (2007). We test how the tail regression approach performs *vis-à-vis* the threshold regression approach.

The setup of the simulations is as follows. In each sample, we generate 2,000 observations

for the series R^m and ε from a student-t distribution with degrees of freedom α . The student-t distribution is known to have heavy tails with the tail index equal to the degrees of freedom. The asset returns are constructed from different linear models. Firstly, we consider the regular linear model with $\beta = 0.5, 0.75, 1$. The β^T coefficient is thus equal to the regular beta. We take $\alpha = 3, 4, 5$. Thus, there are nine models in this category. Secondly, we consider a segmented linear model as follows. If the loss on the market does not exceed a threshold with tail probability 2.5%, the asset return is generated from a regular linear model with $\beta = 0.6$; otherwise, it is generated from the linear model in the tail as in (2.1) with $\beta^T = 0.5, 0.75, 1$ respectively. Within a 2,000 observation sample, about 2,000 * 2.5% = 50 asset returns are generated from the linear relation in the tail. We take $\alpha = 4$. There are three models in this category. For the twelve models in total, we generate 10,000 samples, and estimate β^T from the tail regression and the threshold regression respectively in each sample. Then, by comparing the estimates to the real β^T value, we calculate the mean squared error (MSE) for the two approaches.

- INSERT TABLE 1 -
- INSERT FIGURE 1 -

Figure 1 reports the MSEs with respect to different numbers of high order statistics used in the estimation, k. In addition, Table 1 reports the ratio between the MSEs of from the threshold regression and the tail regression. Values above (below) one demonstrate a better (worse) performance of the tail regression. From Figure 1 and Table 1, we observe a better performance of the tail regression relative to the threshold regression, as one chooses less observations corresponding to more severe losses in estimation (for example, using 1 or 2 percent of the total number of observations). For the nine models, the threshold regression yields a better estimator if many observations are included in the estimation. This is due to the fact that the linear relation in the moderate level does not change in the tail region. Nevertheless, our tail regression approach does not impose a large error even if we include more observations from the moderate level. In the simulations with the segmented model, the linear relation in the moderate level is violated by the observations in the tail. The MSE of the threshold regression can increase strongly when including observations outside the tail region, whereas that of the tail regression is less sensitive to the choice of the number of observations used in estimation, k. Because it is hard to justify where the tail region starts in real data, the tail regression approach would be favored compared to the threshold regression.

3 Empirical Results

We estimate the tail regression beta for 46 equal weighted industry portfolios including NYSE, AMEX and NASDAQ firms. The dataset is available over a long period from the website of Kenneth French.⁷ We collect daily returns for the industry portfolios, the market portfolio and the risk free rate. We give an example of the estimation procedure based on 2,000 daily observations (from 19 July 1999 until 29 June 2007) and then extend the analysis to a moving window setup in order to explore the developments in the time dimension.

- INSERT TABLE 2 -

Table 2 reports descriptive statistics for the individual industry portfolios. *Coal* is perceived as the portfolio with the highest standard deviation in daily returns, while *Banking* has the lowest standard deviation. We observe positive excess kurtosis for all

⁷Web address: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. There are 48 industry portfolios in the dataset. We exclude *Healthcare*, because it is not available over the entire period. We also exclude *Gold*. Gold is known to act as a safe heaven and thus may violate the assumption that $\beta^T \ge 0$. This conjecture is confirmed by observing a large number of observations in the second quadrant of a scatterplot between the returns of the *Gold* portfolio (y-axis) and the market returns (x-axis). Moreover, empirical analysis shows that *Gold* exhibits a negative $\hat{\beta}^{DS}$ and a $\hat{\tau}$ that is not significantly different from 0. Nevertheless, our main conclusions do not depend on either including or excluding *Gold*.

portfolios. This observation confirms a general result from empirical literature that financial return distributions exhibit fatter tails than the normal distribution.

The last two columns in Table 2 report the estimated tail index and scale for the downside tail of the return distribution. The number of tail observations used in estimations, k, equals 60.⁸ As the tail index of the market α_m is estimated at 4.4, the tail index of the individual portfolios α is either below or close to the market tail index. In correspondence with the linear model in the tail, the portfolio α is never significantly higher than α_m from the market portfolio. Although scale plays an important role in analyzing tail risks, comparing scales is only meaningful in case of equal tail indices. For example, *Guns* is riskier than *Automobiles*, as both portfolios have equal tail indices while the higher scale of *Guns* indicates a higher level of tail risk.

- INSERT TABLE 3 -

- INSERT FIGURE 2 -

As the measure of systematic downside risk, downside beta might shed more light on the systematic tail risk than the regular beta. We estimate all three betas, reported in Table 3, and compare them by making scatterplots as in Figure 2. Panel (a) in Figure 2 shows that the downside beta of the industry portfolios is only marginally different from the regular beta. The Pearson and Spearman rank correlation between downside beta and regular beta are both above 0.95. Panel (b) in Figure 2 shows that the tail regression deviates strongly from the regular beta for some portfolios. The dispersion between the regular beta and the tail regression beta is shown by a relatively low Pearson correlation coefficient of 0.76. The low Spearman rank correlation of 0.61 further confirms that regular beta provides a poor ranking of portfolios' systematic tail risk. Moreover,

⁸In extreme value analysis it is always crucial step to choose the number of high order statistics in the estimation procedure, k. We plot $\hat{\beta}^T$ against different k and choose k = 60 which balances the variance and bias. Compared to the sample size (2,000), this includes the 3% worst days.

the Pearson and Spearman rank correlation between the tail regression beta and the downside beta is comparable to that between the tail regression beta and the regular beta. The values are respectively 0.75 and 0.58. To summarize, there can be considerable dispersion between the tail regression beta and the regular beta. Moreover, we find no evidence that downside beta is a better indicator for systematic tail risk than the regular beta. Hence, for risk management purposes, assessing the tail regression beta is necessary.

- INSERT FIGURE 3 -

To assess the relation between the three betas over time, we estimate the betas starting from 19 July 1971 until 31 December 2009 using a backward looking window consisting of 2,000 days. Figure 3 reports the regular beta and the tail regression beta for *Automobiles* and *Ships.*⁹ Panel (a) in Figure 3 suggests that the regular beta for *Automobiles* has been a reasonable good indicator for the tail regression beta. In contrast, Panel (b) in Figure 3 shows that the regular beta for *Ships* has been structurally different from its tail regression beta. Remarkably, the gap between the regular beta and the tail regression beta increased for both portfolios during the stock market boom at the end of the 20th century. This last observation can be generalized to most of the industry portfolios.

- INSERT FIGURE 4 -

Figure 4 reports the cross-sectional correlations from the moving windows analysis between the regular beta and respectively the tail regression beta and the downside beta.¹⁰ The figure illustrates two findings. First, downside beta is highly correlated to regular beta

⁹The two chosen figures help to illustrate different possibilities in the time variation of tail regression beta. The figures for the other portfolios are in general similar to one of the reported figures, and are available upon request.

¹⁰For the sake of clarity we left out the correlation between the tail regression beta and the downside beta, as it is very close to that between regular beta and the tail regression beta over the entire sample period.

over the entire period, whereas the tail regression beta is not. Hence, the finding that the downside beta does not provide a better ranking than regular beta with respect to the industry portfolios' systematic tail risk is robust over time. Second, the dispersion between the regular and the tail regression beta is especially high for moving windows ends from 1992 until 2008, as the Spearman rank correlation is below 0.7. Hence, whether a regular beta can be regarded as an indicator for the systematic tail risk seems to depend on the time window.

Our estimation methodology requires stationarity of the return data. Daily returns are known to be subject to time-wise weak dependence, such as volatility clustering, whereas data with lower frequency suffer less and appear more consistent with the stationarity assumption. To test the robustness of the tail regression beta, we employ data with lower frequency such as weekly observations and compare the results with that from daily observations. To ensure a sufficient number of weekly observations we use return data starting from 1 July 1963 until 31 December 2009. This results in a data sample with 11,707 daily observations or 2,427 weekly observations.

- INSERT TABLE 4 -

Table 4 reports the results from the robustness test for changes in data frequency. The first two rows report the cross-sectional average and standard deviation of all three beta measures as estimated from weekly data. We compare these numbers with the numbers in the third and fourth rows, which contain estimates from daily data. The mean level of the tail regression betas has the smallest change compared to that of the other two betas (from 0.88 for daily data to 0.96 for weekly data), while the change in dispersion is negligible. Hence, the change in the mean level of the tail regression beta and its dispersion is relatively small under the change of data frequency.

To verify whether the tail regression betas estimated from different frequencies present

similar cross-sectional differences, we report the Pearson and Spearman rank correlation between daily and weekly estimates in the fifth and sixth row of Table 4. The correlation coefficients between the tail regression betas are close to 0.9 and have a similar magnitude as that between the downside betas.

In our analysis we intended to explore the development in time. Hence, it is necessary to have a relatively short time with a sufficient number of observations which leaves daily frequency as a reasonable choice. The results in Table 4 raise confidence that the tail regression beta is not sensitive to the choice of data frequency, given that the number of observations is sufficient.

4 Application of tail regression beta

4.1 Portfolio risk management

The analysis on the tail regression beta decomposes the tail risk of assets into a systematic and an idiosyncratic component. Risk managers face the problem of assessing the tail risk of portfolios consisting of multiple assets. Under the linear model in the tail, this assessment is straightforward. Moreover, investors face an alternative problem that is to construct portfolios with a high expected return and limited tail risk. The tail regression beta is again a useful tool in this portfolio optimization problem. In this subsection we demonstrate the application of the tail regression beta for portfolio risk management.

With the heavy-tail feature of the asset return given in (2.3), it is not difficult to obtain that

$$VaR_e(p) \sim \left(\frac{A}{p}\right)^{1/\alpha}, \text{ as } p \to 0.$$
 (4.1)

Hence, with the estimates of the tail index and the scale, the calculation of VaR is straightforward.

Consider d assets with tail regression beta, $\beta_1^T, \dots, \beta_d^T$, and scale of the idiosyncratic

risk $A_{\varepsilon,1}, \dots, A_{\varepsilon,d}$. To simplify the discussion, we consider the case that $\alpha_m = \alpha_{\varepsilon}$. Such an assumption is partially confirmed in our empirical analysis.¹¹ Then $\alpha = \alpha_m = \alpha_{\varepsilon}$. Consider a portfolio consisting of the assets with non-negative weights (w_1, \dots, w_d) such that $\sum_{i=1}^d w_i \leq 1$. Then the portfolio excess return is given by

$$R_P^e = \left(\sum_{i=1}^d w_i \beta_i^T\right) R^m + \sum_{i=1}^d w_i \varepsilon_i.$$

It is clear that also R_P^e has a heavy tail distribution with tail index α . Moreover, the tail regression beta of R_P^e is given by $\beta_P^T = \sum_{i=1}^d w_i \beta_i^T$. From the Feller theorem, the scale of the idiosyncratic risk component is $\sum_{i=1}^d w_i^{\alpha} A_{\varepsilon,i}$. Therefore, following (2.4), we get the scale of the portfolio as

$$A_P = \left(\sum_{i=1}^d w_i \beta_i^T\right)^{\alpha} A_m + \sum_{i=1}^d w_i^{\alpha} A_{\varepsilon,i}.$$
(4.2)

Together with formula (4.1), it is straightforward to calculate the VaR of a portfolio at any low probability level p. Regarding the diversification effect, since the sensitivity of the portfolio to the systematic risk is a weighted average of the tail regression betas of the individual assets, it can not be diversified away. In contrast, since the individual idiosyncratic risks are independent, diversification may be beneficial in terms of lowering the tail risk. For a detailed discussion on the diversification effect with a single factor model in the tail, see Zhou (2010).

Next, we consider the optimal portfolio construction. Constructing a portfolio under safety-first criterion requires controlling for the tail risk measured by $VaR_P(p)$. Following (4.1), this is equivalent to controlling the scale A_P in (4.2) for a fixed probability level pand a given α . It is not difficult to verify that for $\alpha > 1$, the right hand side in (4.2) is a strict convex function with respect to (w_1, w_2, \dots, w_d) . Taking the scale as the objective function, there exists a unique optimal solution according to convex optimization theory. An

¹¹If $\alpha_m > \alpha_{\varepsilon}$, we have $\tau = 0$; if $\alpha_m < \alpha_{\varepsilon}$, we have $\tau = 1$. Since our empirical estimates of τ lie in between 0 and 1. This is regarded as an evidence for the equivalence of the two tail indices.

alternative objective is to maximize the expected return while keeping the tail risk below a given level. This alternative objective is equivalent to a constrained convex optimization problem, which again has a unique optimal solution due to the strict convexity of the scale calculation. To summarize, modeling and estimating the tail regression beta is an important step to simplify the calculation of the aggregated portfolio tail risk, and helps to construct a optimal portfolio with requirements on tail risk.

4.2 Systemic risk management

Under the linear model in the tail, asset returns suffer from extreme market shocks, however, in proportion to their tail regression betas. Hence, the VaR of an asset depends on the market condition. Such a behavior is investigated in a general framework under the name Conditional Value-at-Risk (CoVaR) as in Adrian and Brunnermeier (2010).

The CoVaR of an asset is the VaR of the asset conditional on the fact that the market loss equals to the VaR of the market return. Specifically, for a low probability p, the CoVaR of an excess asset return at probability level p, $CoVaR_e(p)$, is defined by

$$\Pr(R^e < -CoVaR_e(p)|R^m = -VaR_m(p)) = p.$$

$$(4.3)$$

To examine the systemic risk of an asset by CoVaR, two potential comparisons are considered in Adrian and Brunnermeier (2010) and its earlier versions: either comparing the CoVaR measure to the unconditional VaR or comparing the CoVaR measure with a low probability p to the CoVaR measure conditional on the market median. We follow the former method since the calculation of the unconditional VaR is straightforward under our model.¹² The higher the increment from VaR to CoVaR, the more systemic impact one asset suffers from market shocks.

With the linear model in (2.1), the CoVaR calculation is straightforward. Notice that

¹²The comparison to the CoVaR measure conditional on the market median is beyond the reach of the linear model framework, because the model focuses in the tail region only.

given $R^m = -VaR_m(p)$, the only random component in the excess asset return R^e is the idiosyncratic risk ε . Hence,

$$CoVaR_e(p) = \beta^T VaR_m(p) + VaR_e(p).$$
(4.4)

Under the heavy tailed setup in (2.2), the calculation can be more explicit: as $p \to 0$,

$$CoVaR_e(p) \sim \beta^T \left(\frac{A_m}{p}\right)^{1/\alpha_m} + \left(\frac{A_\varepsilon}{p}\right)^{1/\alpha_\varepsilon}.$$
 (4.5)

This formula can be used to estimate the CoVaR measure empirically. Next, we turn to the theoretical comparison between the CoVaR measure and the unconditional VaR. The following theorem shows the comparison result. The proof is in Appendix A.

Theorem 4.1 Suppose the single factor model in (2.1) and the heavy-tail setup of the downside distributions (2.2) and (2.3) hold. With the τ measure defined in (2.5), we have that

$$\lim_{p \to 0} \frac{CoVaR_e(p)}{VaR_e(p)} = \tau^{1/\alpha_m} + (1-\tau)^{1/\alpha_m}.$$

From Theorem 4.1, we get that given a low tail probability p, the CoVaR measure is approximated by multiplying the unconditional VaR measure with a correction factor. The correction factor is a function of the τ measure, which measures the tail dependence between the market return and the asset return. This method can be regarded as an approximate but faster algorithm to estimate CoVaR.

Since the τ measure lies in between 0 and 1, the magnitude of the correction factor is bounded by

$$1 \le \tau^{1/\alpha_m} + (1-\tau)^{1/\alpha_m} \le 2^{1-1/\alpha_m}.$$

The first equality holds if and only if $\tau = 0$ or $\tau = 1$. The second equality holds if and only if $\tau = 1/2$. We conclude that the CoVaR measure is always higher than or equal to the unconditional VaR measure under the linear model in the tail. This is due to the fact that the linear model in the tail imposes tail dependence. With our estimate, $\hat{\alpha}_m = 4.4$, the upper bound $2^{1-1/\alpha_m}$ equals to 1.71. Hence the CoVaR measure is at most 70% higher than the VaR measure for the industrial portfolios in our dataset.

We remark that the correction factor is not an monotone function with respect to τ : it increases on [0, 1/2], while decreases on [1/2, 1]. On the contrary, τ is an increasing function with respect to β^T . Hence, a high systemic risk indicated by the ratio between CoVaR and VaR does not necessarily imply a high systematic risk as indicated by β^T . For example, in both cases $\tau = 0$ and $\tau = 1$, the correction factor equals 1, which indicates that the CoVaR measure is equal to the VaR measure. However, these are the two most opposite cases with respect to tail dependence. We interpret it as follows. If the tail risk of the asset return is independent from that of the market, then a shock in the market return does not change the VaR measure of the asset return. Thus the CoVaR measure is equal to the VaR measure. If the tail risk of the asset return is completely dependent on that of the market, then a shock in the market return is exactly reflected in the asset return movement with proportion β^{T} . Because the asset return is fully determined by that shock and reaches its VaR level, there should be no extra random component. Hence, the CoVaR measure is also equal to the VaR measure in the completely dependent case. By comparing the increment from VaR to CoVaR, it is not possible to distinguish between the two cases. In all, the comparison shows the difference between analyzing the systemic risk and the systematic risk. Hence, in order to have a complete view, it is necessary to investigate tail regression beta alongside the CoVaR analysis.

5 Concluding Remarks

To measure an asset's sensitivity to large negative market shocks, we developed a measure of systematic tail risk, the *tail regression beta*. Building on EVT, we provided a methodology to estimate the tail regression beta. When applying this methodology to industry portfolios, considerable dispersion between the regular market beta and the tail regression beta is observed. Therefore, in order to assess the sensitivity to systematic tail risk, it is recommended to evaluate the tail regression beta rather than directly considering the regular market beta. Furthermore, we demonstrate applications of the tail regression beta in both portfolio risk management and systemic risk management: analyzing VaR and CoVaR.

As an alternative to tail regression in systemic risk management, Adrian and Brunnermeier (2010) have applied quantile regression for CoVaR estimation. Quantile regression is based on modelling the relation between the conditional quantile of the dependent variable and its covariates, see Koenker and Hallock (2001)¹³. Thus quantile regression addresses the sensitivity of quantiles to co-variates. The two approaches to CoVaR differ in model assumption. The tail regression approach models the linear relation between the asset returns and market returns in the tail. The quantile regression approach models the linear relation between the conditional quantile of the asset return and the market return over the entire range of market returns.

Assessing CoVaRs at different probability levels by quantile regression requires repeating the estimation procedure. This is due to the fact that the linear relations modelled by quantile regression are sensitive to the probability levels that define the quantiles. Hence, from estimated linear relations at fixed probability levels, extrapolations towards other probability levels are not possible. Instead, the tail regression model assumes that the tail regression beta remains at the same level given the occurrence of extremely adverse market conditions. This assumption facilitates the possibility to perform CoVaR analysis at levels where observations are rather scarce. To summarize, the quantile regression approach is designed to analyze CoVaR at fixed moderate probability levels, whereas it is possible with tail regression to analyze CoVaR at an extremely low probability level.

Estimating the tail regression beta can be the departing point for research in several other fields. From a corporate finance point of view it may be interesting to find firm

¹³To a certain extent, quantile regression is comparable with the aforementioned threshold regression. This has been discussed in Koenker and Hallock (2001), page 147.

characteristics that explain the cross section of the individual firm's sensitivity to systematic tail risk. Another interesting research direction is to examine whether the spread of the tail regression beta over the regular market beta is priced in the market. Remarkably, we find a significant negative correlation of -0.51 between this spread and the market beta, which raises the conjecture that low beta stocks are more sensitive to systematic tail risk than indicated by the regular betas. These topics are left for future research.

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Appendix: Proofs

Proof of Proposition 2.1

In the case $\beta^T = 0$, we have that R^m and R^e are tail independent, thus $\tau = 0$. The equation (2.5) holds automatically.

In the rest of the proof, we consider $\beta^T > 0$ in the tail regression model. The proof departures from comparing the marginal VaRs of the excess return and the market return in the following lemma.

Lemma A.1 Under the single factor model in (2.1) and the heavy-tail setup of the downside distributions (2.2), we have that, for sufficiently low probability p,

$$\beta^T VaR_m(p) \le VaR_e(p). \tag{A.1}$$

Proof of Lemma A.1

As $p \to 0$, the VaR of the market return $VaR_m(p)$ converges to infinity. Thus, when the tail probability p is sufficiently low such that $VaR_m(p)$ exceeds the threshold in the tail regression model, the linear relation (2.1) is valid for $R^m < -VaR_m(p)$. Hence, we have that for any $\delta > 0$,

$$\Pr(R^{e} < -\beta^{T} VaR_{m}(p)) \ge \Pr(R^{e} < -\beta^{T} VaR_{m}(p) \text{ and } R^{m} < -VaR_{m}(p))$$
$$= \Pr(\beta^{T}R^{m} + \varepsilon < -\beta^{T} VaR_{m}(p) \text{ and } R^{m} < -VaR_{m}(p))$$
$$\ge \Pr(\beta^{T}R^{m} < -\beta^{T} VaR_{m}(p)(1+\delta) \text{ and } \varepsilon < \delta\beta^{T} VaR_{m}(p))$$
$$= \Pr(R^{m} < -VaR_{m}(p)(1+\delta)) \Pr(\varepsilon < \delta\beta^{T} VaR_{m}(p)).$$

The last equality is due to the independency between \mathbb{R}^m and ε . From the heavy-tail setup, we obtain that

$$\lim_{p \to 0} \frac{\Pr(R^m < -VaR_m(p)(1+\delta))}{\Pr(R^m < -VaR_m(p))} = (1+\delta)^{-\alpha_m}.$$

Moreover, it is obvious that $\lim_{p\to 0} \Pr(\varepsilon < \delta \beta^T V a R_m(p)) = 1$. Thus, we have that

$$\liminf_{p \to 0} \frac{\Pr(R^e < -\beta^T VaR_m(p))}{p} \ge (1+\delta)^{-\alpha_m}.$$

Notice that the above inequality holds for all $\delta > 0$. By taking $\delta \to 0$, we get that

$$\liminf_{p \to 0} \frac{\Pr(R^e < -\beta^T VaR_m(p))}{p} \ge 1.$$

From the definition of VaR, $\Pr(R^e < -VaR_e(p)) = p$. Hence, for sufficiently low probability p, inequality (A.1) holds. \Box

Based on the tail regression model (2.1), we calculate the tail dependence measure τ . When the tail probability p is sufficiently low such that $VaR_m(p)$ exceeds the threshold in the tail regression model, a joint tail event can be written as

$$C := \{ R^e < -VaR_e(p) \text{ and } R^m < -VaR_m(p) \}$$
$$= \{ \beta^T R^m + \varepsilon < -VaR_e(p) \text{ and } R^m < -VaR_m(p) \}$$

The τ measure is given as the limit $\lim_{p\to 0} \frac{\Pr(C)}{p}$.

Define $C_0 = \{\beta^T R^m < -VaR_e(p)\}$. We shall show that $\Pr(C) \sim \Pr(C_0)$ as $p \to 0$. For any $0 < \delta < 1$, consider two sets C_1 and C_2 defined as

$$C_1 := \left\{ \beta^T R^m < -VaR_e(p)(1+\delta) \text{ and } \varepsilon < \delta VaR_e(p) \right\},$$

$$C_2 := C_{21} \bigcup C_{22}$$

$$:= \left\{ \beta^T R^m < -VaR_e(p)(1-\delta) \right\} \bigcup \left\{ \varepsilon < -\delta VaR_e(p) \text{ and } R^m < -VaR_m(p) \right\}.$$

It is obvious that $C \subset C_2$. Moreover, from Lemma A.1, we get that $VaR_e(p)(1 + \delta) > VaR_e(p) \geq \beta^T VaR_m(p)$, which implies that $C_1 \subset C$. Hence Pr(C) is bounded by $Pr(C_1)$

and $Pr(C_2)$. We calculate the two probabilities as follows.

Since \mathbb{R}^m and ε are independent, we get that as $p \to 0$,

$$\Pr(C_1) = \Pr(\beta^T R^m < -VaR_e(p)(1+\delta)) \Pr(\varepsilon < \delta VaR_e(p)) \sim \Pr(C_0)(1+\delta)^{-\alpha_m}.$$

Here we use the fact that as $p \to 0$, $\Pr(\varepsilon < \delta VaR_e(p)) \to 1$ and the heavy-tail property of the downside distribution of R^m . It implies that

$$\liminf_{p \to 0} \frac{\Pr(C)}{\Pr(C_0)} \ge (1+\delta)^{-\alpha_m}.$$
(A.2)

Similarly, we get that

$$\lim_{p \to 0} \frac{\Pr(C_{21})}{\Pr(C_0)} = (1 - \delta)^{-\alpha_m}$$

For the set C_{22} , we have that

$$\limsup_{p \to 0} \frac{\Pr(C_{22})}{\Pr(C_0)} = \limsup_{p \to 0} \frac{\Pr(R^m < -VaR_m(p))\Pr(\varepsilon < -\delta VaR_e(p))}{\Pr(C_0)}$$
$$= \limsup_{p \to 0} \frac{p \cdot A_{\varepsilon} \left(\delta VaR_e(p)\right)^{-\alpha_{\varepsilon}}}{A_m (VaR_e(p)/\beta^T)^{-\alpha_m}}$$
$$= \limsup_{p \to 0} \frac{A_{\varepsilon}}{A_m \delta^{\alpha_{\varepsilon}} \beta^{\alpha_m}} p(VaR_e(p))^{\alpha_m - \alpha_{\varepsilon}}$$
$$= 0.$$

The last step comes from the fact that $\alpha_m \leq \alpha_{\varepsilon}$. Therefore,

$$\limsup_{p \to 0} \frac{\Pr(C)}{\Pr(C_0)} \le \limsup_{p \to 0} \frac{\Pr(C_{21})}{\Pr(C_0)} + \frac{\Pr(C_{22})}{\Pr(C_0)} = (1 - \delta)^{-\alpha_m}$$
(A.3)

Since inequalities (A.2) and (A.3) hold for any $0 < \delta < 1$, by combining the two and taking $\delta \to 0$, we get that $\lim_{p\to 0} \frac{\Pr(C)}{\Pr(C_0)} = 1$. Thus,

$$\tau = \lim_{p \to 0} \frac{\Pr(C)}{p} = \lim_{p \to 0} \frac{\Pr(C_0)}{\Pr(R^m < -VaR_m(p))} = \lim_{p \to 0} \left(\frac{\beta^T VaR_m(p)}{VaR_e(p)}\right)^{\alpha_m}.$$

The rest of the proposition follows from straightforward calculation. \Box

Proof of Proposition 2.2

The proof of the equation (2.9) follows exactly the same lines as in the proof of the equation (2.5). The only difference is on the calculation of $\Pr(C_{22})$. To ensure that $\limsup_{p\to 0} \Pr(C_{22}) / \Pr(C_0) = 0$ is still valid in the case $\alpha_m > \alpha_{\varepsilon}$, it is necessary to have a more detailed calculation on the term $\lim_{p\to 0} p(VaR_e(p))^{\alpha_m - \alpha_{\varepsilon}}$. Notice that, when $\alpha_m > \alpha_{\varepsilon}$, $VaR_e(p) \sim VaR_{\varepsilon}(p) = O(p^{-1/\alpha_{\varepsilon}})$. Hence,

$$p(VaR_e(p))^{\alpha_m - \alpha_{\varepsilon}} = pO\left(p^{-1/\alpha_{\varepsilon}}\right)^{\alpha_m - \alpha_{\varepsilon}} = O\left(p^{2-\alpha_m/\alpha_{\varepsilon}}\right).$$

Since we assume $\alpha_{\varepsilon} < \alpha_m < 2\alpha_{\varepsilon}$, the power $2 - \alpha_m/\alpha_{\varepsilon}$ is positive. Therefore, we still have that $\limsup_{p\to 0} \Pr(C_{22})/\Pr(C_0) = 0$. The rest of the proof is similar. \Box

Proof of Theorem 2.3

In the case $\alpha_m \leq \alpha_{\varepsilon}$ and $\beta^T > 0$, τ is a finite positive number. By inverting (2.5), we get that

$$\beta^T = \tau^{1/\alpha_m} \lim_{p \to 0} \frac{VaR_e(p)}{VaR_m(p)}$$

For a intermediate sequence k := k(n) such that $k \to \infty$ and $k/n \to 0$ as $n \to \infty$, we have that

$$\lim_{p \to 0} \frac{VaR_e(p)}{VaR_m(p)} = \lim_{n \to \infty} \frac{VaR_e(k/n)}{VaR_m(k/n)}$$

Since an order statistic is a non-parametric estimate of the VaR, we have that $X_{n,n-k}/VaR_e(k/n) \xrightarrow{P} 1$ and $X_{n,n-k}^{(m)}/VaR_m(k/n) \xrightarrow{P} 1$ as $n \to \infty$. Hence

$$\lim_{n \to \infty} \frac{X_{n,n-k}/X_{n,n-k}^{(m)}}{VaR_e(k/n)/VaR_m(k/n)} = 1, \quad \text{in probability}$$

From the multivariate extreme value statistics, $\hat{\tau}$ is a consistent estimator of τ ; see e.g. Theorem 7.2.1 in de Haan and Ferreira (2006). Together with the consistency of the Hill estimator $\hat{\alpha}_m$, the consistency of $\hat{\beta}^T$ is obvious. In the case that both $\alpha_m \leq \alpha_{\varepsilon}$ and $\beta^T = 0$, we have $\tau = 0$, thus $\hat{\tau} \xrightarrow{P} 0$. Moreover, as $p \to 0, \frac{VaR_e(p)}{VaR_m(p)} = O(1)$, which implies that $\frac{X_{n,n-k}}{X_{n,n-k}^{(m)}} = O_p(1)$. Hence, the consistency of $\hat{\beta}^T$ is still valid.

In the case $\alpha_{\varepsilon} < \alpha_m < 2\alpha_{\varepsilon}, \tau(p) \to 0$ as $p \to 0$. In other words, $\tau = 0$. The consistency of the τ estimator has to be interpreted in a different way as $\frac{\hat{\tau}}{\tau(k/n)} \xrightarrow{P} 1$ as $n \to \infty$. This consistency relation is ensured by the proof of Theorem 7.6.1 in de Haan and Ferreira (2006) (page 267, line 7). The rest of the proof is identical to that in the case $\alpha_m \leq \alpha_{\varepsilon}$. \Box

Proof of Theorem 4.1

We prove the theorem under three cases: $\alpha_m = \alpha_{\varepsilon}, \ \alpha_m < \alpha_{\varepsilon} \ \text{and} \ \alpha_m > \alpha_{\varepsilon}$.

If $\alpha_m = \alpha_{\varepsilon}$, from (4.5), we get that, as $p \to 0$,

$$CoVaR_e(p) \sim \frac{\beta^T A_m^{1/\alpha_m} + A_{\varepsilon}^{1/\alpha_m}}{p^{1/\alpha_m}}$$

and

$$VaR_e(p) = \frac{A^{1/\alpha_m}}{p^{1/\alpha_m}}.$$

Hence,

$$\lim_{p \to 0} \frac{CoVaR_e(p)}{VaR_e(p)} = \frac{\beta^T A_m^{1/\alpha_m} + A_{\varepsilon}^{1/\alpha_m}}{A^{1/\alpha_m}} = \left(\frac{(\beta^T)^{\alpha_m} A_m}{A}\right)^{1/\alpha_m} + \left(\frac{A_{\varepsilon}}{A}\right)^{1/\alpha_m} = \tau^{1/\alpha_m} + (1-\tau)^{1/\alpha_m}.$$

The theorem is proved.

If $\alpha_m < \alpha_{\varepsilon}$, the τ measure has to be either one or zero depending on whether $\beta^T > 0$ or $\beta^T = 0$. In both cases, $\tau^{1/\alpha_m} + (1-\tau)^{1/\alpha_m} = 1$. On the other hand, from (4.5), we have that

$$CoVaR_e(p) \sim \beta^T \left(\frac{A_m}{p}\right)^{1/\alpha_m} \sim VaR_e(p).$$

Hence the theorem holds for the case $\alpha_m < \alpha_{\varepsilon}$.

In the case $\alpha_m > \alpha_{\varepsilon}$, we must have $\tau = 0$, the proof is similar to the case $\alpha_m < \alpha_{\varepsilon}$ with the only difference that the term $\left(\frac{A_{\varepsilon}}{p}\right)^{1/\alpha_{\varepsilon}}$ dominates both the VaR and the CoVaR. \Box

Tables and Figures

	Tail region	$\beta^T = 0.50$	$\beta^T = 0.75$	$\beta^T = 1.00$
	1%	1.7	2.9	3.9
$\alpha = 3$	2%	1.0	1.4	1.7
	3%	0.6	0.8	1.3
	1%	2.1	3.9	5.8
$\alpha = 4$	2%	1.1	1.7	2.5
	3%	0.7	1.0	1.6
	1%	2.3	5.0	7.7
$\alpha = 5$	2%	1.3	2.0	3.2
	3%	0.7	1.2	2.0
Non-linear	1%	1.8	4.4	6.7
	2%	1.2	1.8	3.7
	3%	0.6	1.4	7.3

Table 1: Simulations: comparison between tail regression and threshold regression

Note: The reported numbers are the ratios between the mean squared error of the threshold regression beta and that of the tail regression beta from the estimator in (2.8). The simulations are based on 10,000 samples with 2,000 observations each. Observations are randomly drawn from Student-t distributions with respectively 3, 4 and 5 degrees of freedom. The asset returns in the simulations in the first three rows are based on a linear relation that does not depend on the market conditions (i.e. $\beta^T = \beta$). The asset return in the non-linear model is based on a segmented linear model, where β is set to 0.6 if the loss on the market does not exceed $VaR_m(0.025)$ (i.e. approximately 50 observations in each sample are affected by the linear model in the tail with β^T).

Industry	Mean	St.dev	Skewness	Excess Kurtosis	Tail Index	Scale
Banks	0.06	0.5	-0.1	1.7	3.9	0.02
Food	0.07	0.7	-0.1	1.8	4.3	0.08
Meals	0.08	0.8	-0.2	5.7	3.2	0.10
RlEst	0.08	0.9	0.5	4.4	3.7	0.16
Beer	0.09	1.0	2.1	29.5	3.9	0.17
Insur	0.06	0.8	-0.1	2.4	3.2	0.10
BldMt	0.07	0.9	0.0	1.7	4.4	0.32
Soda	0.06	1.2	0.2	4.8	3.5	0.43
Hshld	0.05	0.9	0.1	2.4	3.8	0.21
Util	0.04	0.9	-0.5	5.4	3.3	0.17
PerSv	0.07	1.0	0.2	3.4	3.7	0.24
Clths	0.07	0.9	-0.2	2.2	4.5	0.38
Ships	0.08	1.3	0.2	1.9	3.6	0.52
Paper	0.03	0.9	-0.1	2.2	4.1	0.29
Fin	0.08	1.0	0.2	2.6	4.4	0.38
Whlsl	0.08	0.9	-0.2	2.2	4.6	0.42
Books	0.03	1.0	0.5	4.0	4.3	0.40
MedEq	0.11	1.0	-0.3	3.4	3.6	0.27
Chems	0.06	1.1	0.0	1.5	4.6	0.60
Rubbr	0.09	1.3	9.1	238.9	4.7	0.72
Trans	0.06	1.1	-0.7	10.3	3.8	0.38
Agric	0.08	1.4	1.3	12.2	2.7	0.23
ElcEq	0.07	1.1	0.0	1.7	4.1	0.55
Rtail	0.06	1.1	-0.1	3.1	3.8	0.37
Mines	0.09	1.5	0.1	1.5	3.3	0.76
BusSv	0.08	1.0	-0.2	3.3	3.8	0.34
Mach	0.08	1.1	0.1	1.6	4.1	0.52
Other	0.09	1.1	-0.1	3.8	3.0	0.23
Toys	0.06	1.1	0.0	2.1	3.8	0.48
Autos	0.03	1.2	0.0	2.9	3.4	0.42
Fun	0.08	1.2	-0.1	3.6	3.3	0.34
Aero	0.10	1.3	0.1	7.7	3.7	0.67
FabPr	0.04	1.5	0.3	2.5	3.6	1.06
Oil	0.12	1.4	-0.2	0.7	4.7	3.29
Cnstr	0.08	1.3	0.0	0.9	4.0	0.95
Txtls	0.03	1.4	0.3	3.5	3.4	0.65
Steel	0.07	1.4	-0.1	1.2	4.1	1.30
Guns	0.14	1.8	2.9	28.7	3.4	0.97
Smoke	0.10	1.8	1.8	30.0	3.4	1.10
LabEq	0.10	1.3	0.0	2.5	4.5	2.03
Boxes	0.04	1.5	0.4	3.0	3.6	1.17
Drugs	0.10	1.5	0.0	5.2	3.2	0.84
Hardw	0.08	1.7	0.2	4.5	3.9	2.41
Coal	0.12	2.0	0.3	3.6	3.7	3.38
Telcm	0.01	1.6	0.4	4.6	4.1	3.27
Chips	0.09	1.9	0.2	3.1	4.0	3.90

Table 2: Descriptive Statistics of the excess returns from 46 industry portfolios

Note: Estimations are based on daily excess return data of 46 industry portfolios from 19 July 1999 until 29 June 2007. The tail index and scale are estimated by the Hill estimator in (2.6) based on the 3% lowest returns.

Table 3: Beta Estimates

Industry	Tail Beta	Regular Beta	Downside Beta	τ	$s.d.(\tau)$	Residual Scale
Banks	0.34	0.38	0.31	0.35	0.06	0.02
Food	0.45	0.38	0.36	0.33	0.06	0.05
Meals	0.51	0.47	0.48	0.30	0.05	0.07
RlEst	0.51	0.35	0.34	0.22	0.06	0.13
Beer	0.51	0.29	0.34	0.22	0.05	0.14
Insur	0.56	0.58	0.53	0.48	0.06	0.05
BldMt	0.60	0.60	0.56	0.32	0.06	0.22
Soda	0.60	0.37	0.35	0.12	0.04	0.38
Hshld	0.61	0.59	0.58	0.35	0.06	0.14
Util	0.61	0.47	0.47	0.35	0.06	0.11
PerSv	0.62	0.57	0.64	0.32	0.06	0.16
Clths	0.62	0.57	0.59	0.32	0.05	0.26
Ships	0.65	0.54	0.43	0.13	0.04	0.45
Paper	0.65	0.59	0.57	0.40	0.06	0.17
Fin	0.65	0.70	0.64	0.37	0.06	0.24
Whlsl	0.66	0.62	0.66	0.40	0.06	0.25
Books	0.66	0.59	0.59	0.37	0.06	0.25
MedEq	0.67	0.61	0.64	0.35	0.06	0.18
Chems	0.68	0.69	0.62	0.32	0.06	0.41
Rubbr	0.70	0.50	0.53	0.32	0.06	0.49
Trans	0.70	0.72	0.67	0.35	0.05	0.25
Agric	0.72	0.42	0.51	0.27	0.05	0.17
ElcEq	0.72	0.72	0.74	0.32	0.06	0.38
Rtail	0.72	0.77	0.76	0.40	0.05	0.22
Mines	0.73	0.59	0.49	0.10	0.04	0.68
BusSv	0.73	0.75	0.81	0.47	0.06	0.18
Mach	0.74	0.83	0.78	0.40	0.06	0.31
Other	0.74	0.66	0.74	0.43	0.06	0.13
Toys	0.75	0.56	0.57	0.33	0.06	0.32
Autos	0.78	0.75	0.70	0.35	0.05	0.27
Fun	0.79	0.71	0.74	0.43	0.07	0.19
Aero	0.80	0.61	0.66	0.30	0.05	0.47
FabPr	0.81	0.53	0.48	0.17	0.05	0.89
Oil	0.81	0.54	0.55	0.15	0.05	2.79
Cnstr	0.82	0.76	0.69	0.28	0.06	0.68
Txtls	0.82	0.55	0.48	0.23	0.05	0.50
Steel	0.86	0.84	0.78	0.27	0.05	0.96
Guns	0.87	0.35	0.33	0.18	0.05	0.80
Smoke	0.93	0.49	0.48	0.20	0.05	0.88
LabEq	0.94	0.94	0.97	0.38	0.06	1.25
Boxes	0.98	0.79	0.77	0.33	0.06	0.78
Drugs	1.04	0.99	1.12	0.38	0.05	0.52
Hardw	1.10	1.18	1.23	0.32	0.05	1.65
Coal	1.12	0.68	0.54	0.18	0.05	2.76
Telcm	1.22	1.12	1.19	0.47	0.06	1.75
Chips	1.23	1.34	1.33	0.35	0.06	2.54

Note: Estimations are based on daily excess return data of 46 industry portfolios from 19 July 1999 until 29 June 2007. The τ measure and the tail regression beta are estimated from the estimators in (2.7) and (2.8) based on the 3% lowest returns. The residual scale is calculated from equation (2.4) with assuming $\alpha_m = \alpha_{\varepsilon}$.

		Tail Beta	Regular Beta	Downside Beta
Weekly	Mean	0.96	0.90	1.02
	Standard deviation	0.15	0.15	0.17
Daily	Mean	0.88	0.77	0.84
	Standard deviation	0.15	0.13	0.14
Correlation	Pearson	0.92	0.95	0.90
	Spearman	0.87	0.95	0.90

Table 4: Robustness check: comparison between estimates from weekly and daily returns

Note: Estimations are based on respectively weekly and daily excess return data of 46 industry portfolios from 1 July 1963 until 31 December 2009. The first four rows report the cross-sectional mean and standard deviation. The fifth and sixth row report the cross-sectional correlation between beta estimates from weekly and daily data.



Figure 1: Simulated mean squared errors for tail regression and threshold regression

Number of observations used in estimation, k

Note: The figures show simulated mean squared errors (MSE) with respect to the number of observations used in estimation, k. The solid [dashed] lines report the MSEs of the estimates from the tail regression [threshold regression]. The simulations are based on 10,000 samples with 2,000 observations each. Observations are randomly drawn from Student-t distributions with respectively 3, 4 and 5 degrees of freedom. The asset returns in the simulations in the first three rows are based on a linear relation that does not depend on the market conditions (i.e. $\beta^T = \beta$). The asset return in the non-linear model is based on a segmented linear model, where β is set to 0.6 if the loss on the market does not exceed $VaR_m(0.025)$ (i.e. approximately 50 observations in each sample are affected by the linear model in the tail with β^T).



Figure 2: Downside beta and tail regression beta versus regular beta

Note: The figures report the estimated regular betas of the industry portfolios versus their downside betas (*panel a*) and their tail regression betas (*panel b*) respectively. Estimations are based on daily excess return data of 46 industry portfolios from 19 July 1999 until 29 June 2007. The tail regression beta are estimated from the estimators in (2.7) and (2.8) based on the 3% lowest returns.

Figure 3: The moving window results of beta estimates: Automobiles and Ships



Note: The figures report the moving window estimation results for the regular betas and the tail regression betas of two industry portfolios: *Automobiles* (panel a) and *Ships* (panel b). The solid [dashed] line reports the tail regression beta [regular beta]. The estimations start from 19 July 1971 until 31 December 2009 using a backward looking window consisting of 2,000 days.





Note: The figures report the cross-sectional correlations between the beta estimates from the moving window results. The solid [dashed] line reports the correlation between the tail regression beta [downside beta] and the regular market beta. The estimations start from 19 July 1971 until 31 December 2009 using a backward looking window consisting of 2,000 days.

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