

Deep reinforcement learning in a monetary model

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Central bankers go data driven: applications of AI and ML for policy and prudential supervision

12. & 13. May 2022, De Nederlandsche Bank, Amsterdam

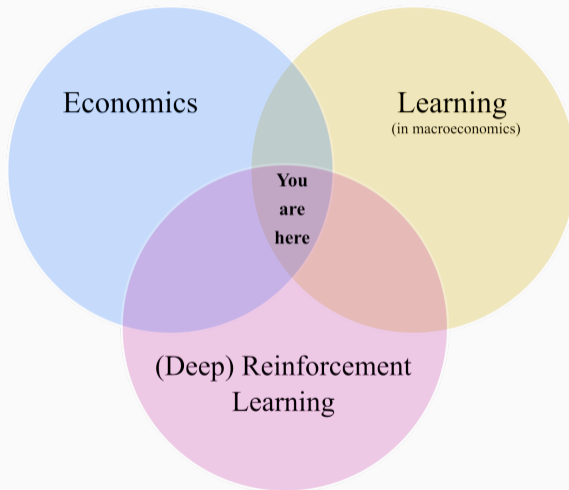
*Disclaimer: The expressed views are our own and should not be represented as those of the Bank of England (BoE). All errors are ours.

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Overview & motivation



- **(Macro)economics:** Focus on **rational expectations** (RE) general equilibrium approaches.
- **Learning agent:** Relaxes RE assumption (next slide)
- **(Deep) reinforcement learning (DRL):** General approach to learning (and much more)

Why learning in (macro)economics?

- RE convenience choice to solve a model, but not necessarily how people and businesses actually behave
- Learning approach to **bounded rationality (BR)**: Specify agent knowledge and behaviour away from RE (often ad hoc)
- BR broadens available state space: RE equilibrium only one possible trajectory
- Example: Monetary policy reaction functions possibly very different under learning, such as forward guidance or the stability of Taylor rules

See Eusepi and Preston (2018) and Hommes (2021) for recent reviews.

Example: Adaptive learning (Evans and Honkapohja, 2001)

Agents are “econometricians” trying to estimate expected quantities

$$\mathbb{E}_t[x_{t+1}] = x_{t+1}^e = x_t^e + \phi_t(x_{t-1} - x_t^e), \quad (1)$$

with a gain series ϕ_t . Together with the (optimal) behavioural rules, i.e. linearised FOCs, this leads to a set of ordinary differential equations determining the expectations (E-)stability of the model.

That is, if a steady state is **stable under learning**, which then serves as a selection criterion.

Applications of DRL in macroeconomics

- **Global solution technique** with no need of linearisation or other approximations
- Principled way to **(bounded) rationality**, i.e. agent behaviour and knowledge (this paper)
- **General approach to heterogeneity**, e.g. household income or age distribution (Hill et al., 2021).

⇒ Loads of potential applications in macro!

Deep reinforcement learning

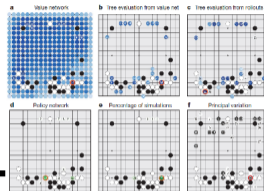
ARTICLE

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Mastering the game of Go with deep neural networks and tree search

David Silver^{1*}, Aja Huang^{1*}, Chris J. Maddison¹, Arthur Guez¹, L. Julian Schrittwieser¹, Ioannis Antonoglou¹, Veda Panneershelvam¹, John Nham², Nal Kalchbrenner¹, Ilya Sutskever², Timothy Lillicrap¹ & Demis Hassabis¹

Playing Atari with Deep Reinforcement Learning



[AUTOMATE](#) [EXPLORE](#) [CUSTOMIZE](#)



STRONG

Carry and power up to 14kg of inspection equipment.



EASY TO CONTROL

Control the robot from afar using an intuitive tablet application and built-in stereo cameras.

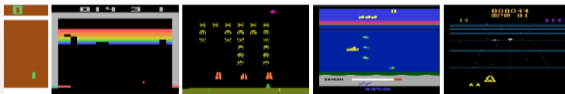


SMART

Program repeatable autonomous missions to gather consistent data.

David Silver^{1*}, Koray Kavukcuoglu¹, David Silver¹, Alex Graves¹, Ioannis Antonoglou¹

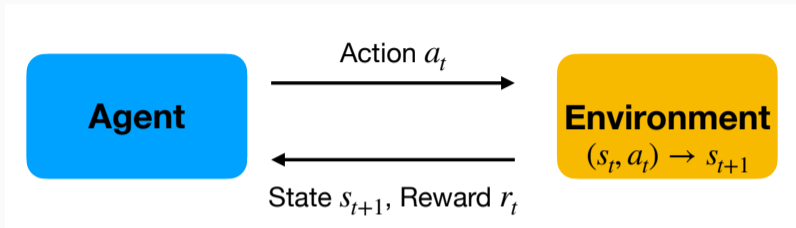
Daan Wierstra¹, Martin Riedmiller¹



Screen shots from five Atari 2600 Games: (Left-to-right) Pong, Breakout, Space Invaders, Endorphin Rider, Asterix

Sources: Nature, arXiv, Boston Dynamics

The reinforcement learning (RL) setting



1. Agent observes state of the world s_t
2. Agent takes actions $a_t(s_t)$
3. Agent receives reward r_t from environment
4. Actions and state lead to state transition of the environment s_{t+1}

This setting is very general. See Sutton and Barto (2018) for a comprehensive introduction.

Formal RL definition

The agent aims to maximise expected cumulative lifetime reward, or **expected return**,

$$\max_{\mathcal{P}} \mathbb{E}_t[G_t] \quad \text{with} \quad G_t \equiv \sum_{k=0}^{\infty} \beta^k r_{t+1+k}(s), \quad (2)$$

following a **behavioural policy** $\mathcal{P} : s_t \rightarrow a_t$, with $s_t \in \mathcal{S} \subset \mathbb{R}^{n_s}$ (state space) and $a_t \in \mathcal{A} \subset \mathbb{R}^{n_a}$ (action space).

The **environment** the agents interaction with returns a reward and a new state, i.e. $\mathcal{E} : (s_t, a_t) \rightarrow (s_{t+1}, r_t)$, with $r_t \in \mathbb{R}$.

The **state transitions** is modelled as a Markov decision process (MDP) $\mathcal{T} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow \text{Pr}(s_{t+1}|s_t, a_t) \in [0, 1]$.

Problem: Writing down \mathcal{T} is simple, knowing $\mathbb{E}_t[G_t]$ and $\text{Pr}(s_{t+1}|s_t, a_t)$ is hard (dynamic programming, value function iteration, etc.).

State and action values

The expected return is maximised by finding the policy \mathcal{P}^* , which maximises the **values function**

$$V_{\mathcal{P}}(s) = \mathbb{E}_{\mathcal{P}}[G_t | s = s_t] \quad (3)$$

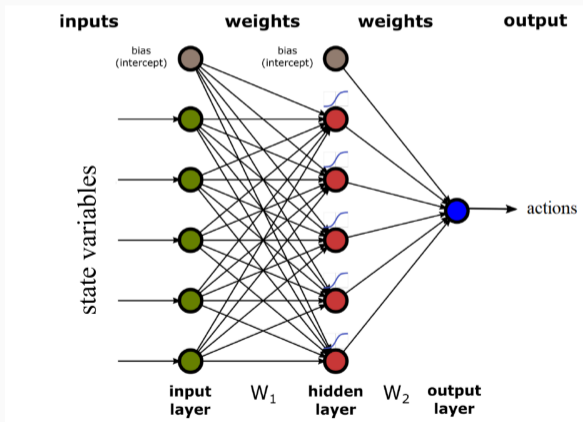
$$\begin{aligned} &= \max_{a \in \mathcal{A}} \mathbb{E}_{\mathcal{P}}[G_t | s = s_t, a = a_t] \\ &= \max_{a \in \mathcal{A}} Q(s, a), \end{aligned} \quad (4)$$

with $Q(s, a)$ the **action-value function**. We are done if we know \mathcal{P}^* and V^*/Q^* .

There are different ways to address this problem, which is an area of active AI research.

Deep reinforcement learning (DRL)

In DRL, functions \mathcal{P} and V/Q are parameterised using **deep artificial neural networks** (Goodfellow et al., 2016), i.e. neural nets with several hidden layers, \mathcal{P}_ϕ and Q_θ :

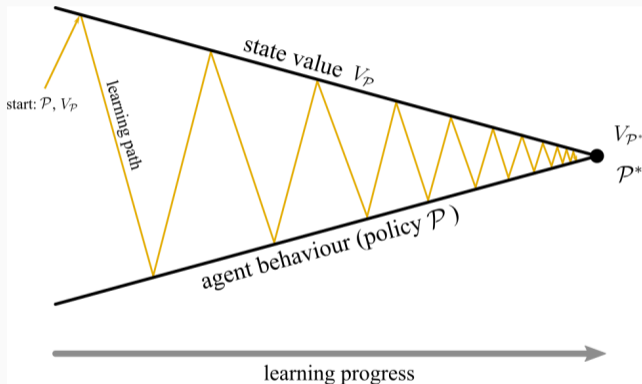


General DRL setting for (macro)economics

- Write down model (environment and state)
- Specify **learning** agents, e.g. households, firms, etc., and their **actions**
- Specify state transitions as MDP
- Learning using DRL algorithm (e.g. Haarnoja et al. (2018)): learning protocol
 1. sample state transition(s) and store in memory
 2. train \mathcal{P}_ϕ and Q_θ from memory
 3. test \mathcal{P}_ϕ and Q_θ with new state transitions and metric of choice

Generalised policy iteration (GPI)

GPI connects economics and learning, and conventional learning approaches with RL



V^* : steady state values, \mathcal{P}^* : FOC.

Examples of RL in economics and finance

- Charpentier et al. (2020): Brief introduction to RL in a economics and finance background
- Zheng et al. (2020): Learning in large-scale geographic ABM
- Calvano et al. (2020): Investigate algorithmic collusion in financial markets
- Chaudhry and Oh (2020): Extract high-frequency expectations in financial markets to measure information effects
- Castro et al. (2021): Learn policy rules of banks participating in a high-value payments system

The model environment

A single representative household maximises its expected lifetime utility, subject to an inter-temporal budget constraint:

$$\max_{c_t, m_t, n_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, m_t, n_t) \quad \text{s.t.} \quad (5)$$

$$M_t + B_t + C_t = M_{t-1} + B_{t-1}R_{t-1} + W_t n_t - P_t \tau_t, \quad (6)$$

with P_t the price level at time t , $x_t = \frac{X_t}{P_t}$, $x \in \{M_t, B_t, C_t, W_t\}$ relate real and nominal money, government bonds, consumption and wages, and τ_t is a real lump-sum tax to the government each period.

We take the utility (Evans and Honkapohja, 2005)

$$U(c_t, m_t, n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \chi \frac{m_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi}. \quad (7)$$

A single representative firm produces according to

$$y_t = \varepsilon_t^y n_t, \quad (8)$$

with technology (shock) ε_t^y , maximising profits

$$\max_{w_t} y_t - w_t n_t, \quad (9)$$

by setting the optimal wage¹

$$w_t = \varepsilon_t^y. \quad (10)$$

¹This could be replaced by DRL resulting in a multi-agent learning problem.

Markets clear every period, i.e.

$$y_t = c_t \quad (\text{goods}), \quad (11)$$

and

$$c_t^\sigma n_t^\varphi = \varepsilon_t^y \quad (\text{labour}). \quad (12)$$

The government issues interest-bearing bonds and non-interest-bearing currency (money), and collects taxes under the real inter-temporal *government budget constraint* (GBC)

$$m_t + b_t + \tau_t = \frac{m_{t-1}}{\pi_t} + R_{t-1} \frac{b_{t-1}}{\pi_t}, \quad (13)$$

subject to the transversality condition

$$\lim_{j \rightarrow \infty} \prod_{k=0}^j \left(\frac{\pi_{t+k}}{R_{t+k-1}} \right) b_{t+j} = 0. \quad (14)$$

Fiscal policy takes the linear tax rule as in Leeper (1991)

$$\tau_t = \gamma_0 + \gamma b_{t-1} + \varepsilon_t^\tau, \quad (15)$$

where ε_t^τ is an exogenous random shock that is assumed to be i.i.d. with mean zero, and $0 \leq \gamma \leq \beta^{-1}$. We follow Leeper (1991) to define fiscal policy as being **active** if $\gamma < \beta^{-1} - 1$ (AFP) and **passive** if $\gamma > \beta^{-1} - 1$ (PFP).

We follow Benhabib et al. (2001) and Evans and Honkapohja (2005) with a global non-linear interest rate rule

$$R_t - 1 = \varepsilon_t^R f(\pi_t) \quad (\textit{Taylor rule}), \quad (16)$$

with $f(\pi)$ assumed to be **non-negative and nondecreasing**, and ε_t^R is an exogenous, i.i.d. and positive random shock with a mean of one:

$$f(\pi_t) = (R^* - 1) \left(\frac{\pi_t}{\pi^*} \right)^{\frac{AR^*}{R^* - 1}}, \quad (17)$$

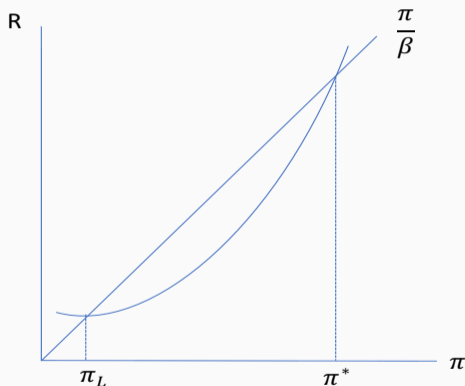
where $A > 1$, and $\pi^* > 1$ is the inflation target.

Steady states

The Taylor rule (16) implies **two steady states** at the intersection with the Euler/Fisher equation

$$\frac{\pi}{\beta} = 1 + (R^* - 1) \left(\frac{\pi}{\pi^*} \right)^{\frac{AR^*}{R^* - 1}}. \quad (18)$$

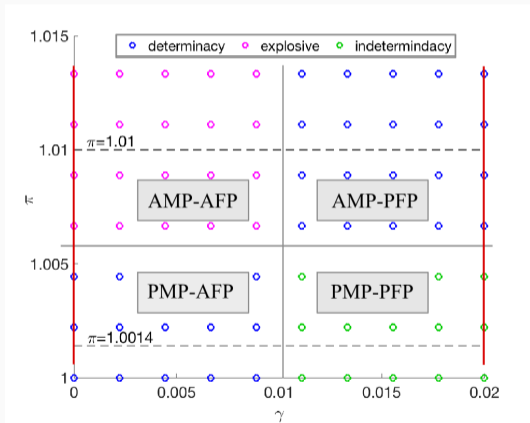
Monetary policy (MP) is said to be **active** at π^* ($f'(\pi_t) > 1$; AMP) and **passive** at π_L ($f'(\pi_t) < 1$; PMP).



This situation is very general and commonly investigated in learning in macroeconomics.

Policy regimes

Using a standard parameterisation and local stability analysis we obtain four policy regimes



	AMP (π^*)		PMP (π_L)	
	PFP	AFP	PFP	AFP
π_{ss}	1.0100	1.0100	1.0014	1.0014
m_{ss}	1.7157	1.7157	2.0614	2.0614
$c_{ss}/n_{ss}/y_{ss}$	1	1	1	1
b_{ss}	4	4	4	4
u_{ss}	-1.0170	-1.0170	-1.0118	-1.0118
γ_0	-0.0566	0.0234	-0.0426	0.0375

Joining the model and RL

State representation

$$s_t = \left(m_{t-1}, b_{t-1}, \pi_{t-1}, c_{t-1}, n_{t-1}, \epsilon_t^\tau, \epsilon_t^R, \epsilon_t^y \right). \quad (19)$$

Household agent actions

$$a_t = (c_t^{act}, b_t^{act}, n_t), \quad (20)$$

where $x_t^{act} = X_t/P_{t-1}$, $x \in \{c, b\}$. Information flow and market clearing

$$\pi_t = c_t^{act} / y_t, \quad (21)$$

$$c_t = c_t^{act} / \pi_t, \quad (22)$$

$$b_t = b_t^{act} / \pi_t. \quad (23)$$

Model environment: Production, market clearing, pricing, GBC, FP, MP.

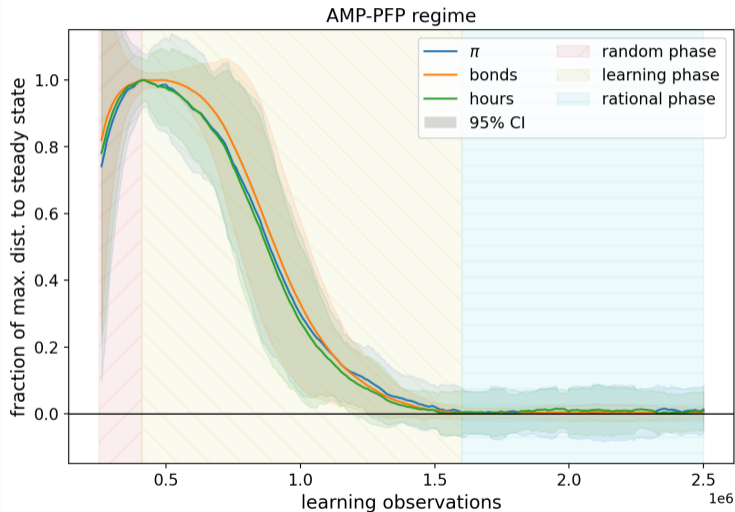
No first-order conditions (FOC)

State transition

1. Observe state s_t
2. Take actions $\mathcal{P}_\phi(s_t) = a_t = (b_t^{act}, c_t^{act}, n_t)$
3. Production (8) takes place and firm sets wages (10)
4. Markets clear: Inflation π_t is set by (21)
5. This determines real consumption c_t and bond holdings b_t (22)-(23)
6. Policy realisations:
 - The monetary authority sets the current gross interest rate R_t via the Taylor rule (16)
 - The government raises taxes τ_t (15)
7. The money holdings m_t are realised from the GBC (13)
8. Agent obtains reward $r_t = U(c_t, m_t, n_t)$
9. Next periods shocks are realised, $(\epsilon_{t+1}^\tau, \epsilon_{t+1}^R, \epsilon_{t+1}^y)$
10. State update $s_t \leftarrow s_{t+1} = (m_t, b_t, \pi_t, c_t, n_t, \epsilon_{t+1}^\tau, \epsilon_{t+1}^R, \epsilon_{t+1}^y)$

Results

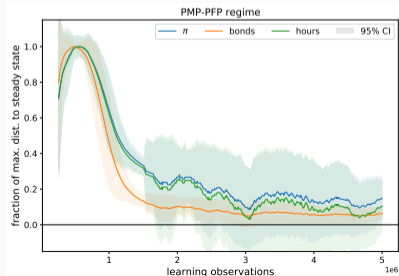
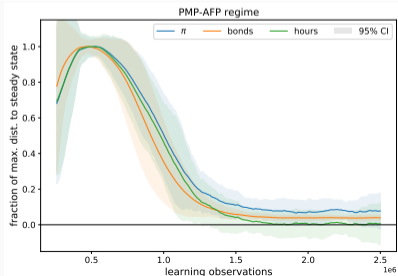
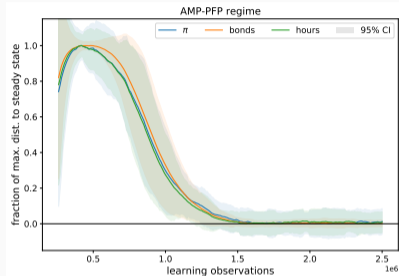
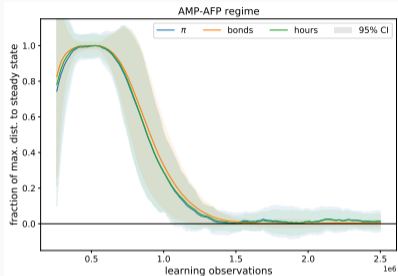
Steady state learning in the AMP-PFP regime



Learning phases

- random (agent initialisation)
- learning
- rational

Steady state learning in all regimes (charts)



Steady state learning in all regimes (table)

	AMP (π^*)		PMP (π_L)	
	PFP	AFP	PFP	AFP
AL	yes	no	no	yes
RL	yes	yes	yes [†]	yes [†]
	$ \Delta_{ss} $ (%) for RL			
π	0.346	0.278	9.217	5.209
b	0.005	0.004	0.038	0.024
n	0.004	0.003	0.009	0.003
m	0.091	0.089	11.569	7.364
u	0.003	0.003	0.346	0.196

[†]imprecision in learning about inflation at π_L .

Measuring bounded rationality

The household is said to behave rational if it follows FOC. During learning, we define the **FOC-distance** to measure deviations in a standardised way

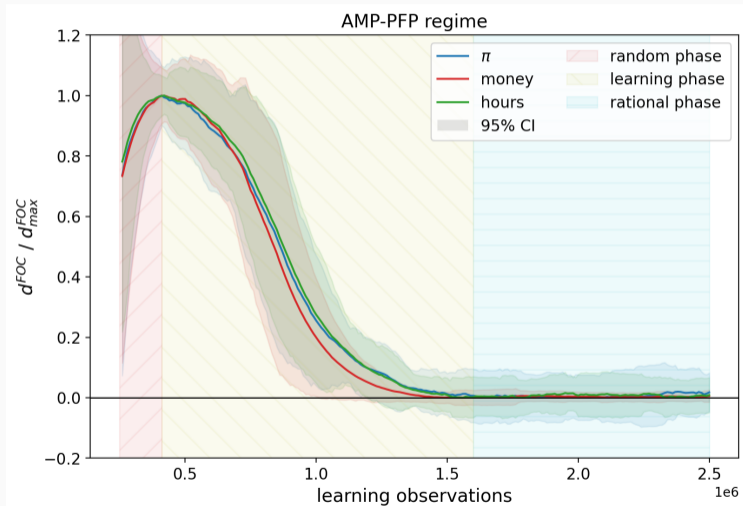
$$d_x^{FOC} \equiv |FOC(x) - 1|, \quad (24)$$

The explicit expression for the **Euler equation**, or *Euler distance*, is

$$d_\pi^{FOC} = \left| \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] - 1 \right|. \quad (25)$$

FOC distances evaluate agent actions $\mathcal{P}(s)$. Analogous measures for V/Q can be derived with respect to state values.

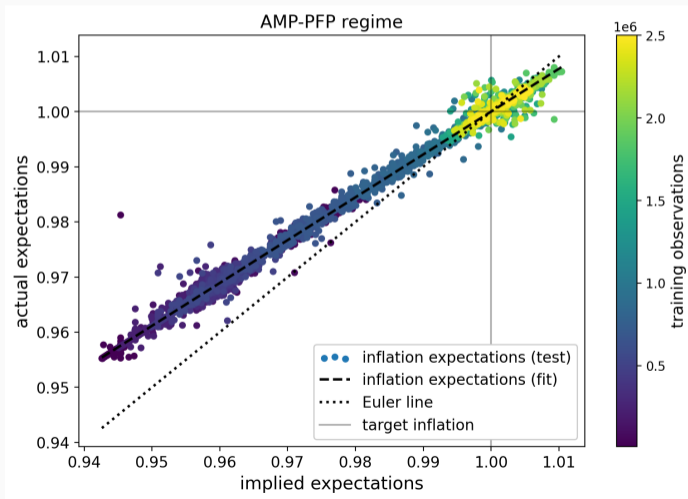
FOC learning in the AMP-PFP regime



The same Learning phases as expected by GPI.

Measuring inflation expectations during learning

Implied agent expectations can be extracted from realised values



- Improve and better understand learning **robustness**
- Aim for truly global learning
- Compare IRF with those from adaptive learning
- Conduct **experiments**: policy or regimes shifts

Take-away messages

- DRL offers a **general approach** to solve structural macro models
- **Quantify bounded rationality** and learning in a principled way: agent behaviour as a free model parameter
- **Global** solution techniques which can also address heterogeneity
- **All policy regimes** are learnable under DRL
- **Promising toolbox** for (macro)economics
- Learning and convergence **technically challenging**

Thanks for listening

Q& A

Actor-critic DRL setting

\mathcal{P} and Q fulfil the **Bellman equation**

$$Q(s_t, a_t) = r(s_t) + \beta \mathbb{E}_{\mathcal{P}} [Q(s_{t+1}, a_{t+1})]. \quad (26)$$

using sampled state transitions as observations, i.e. interactions of the agent and the environment, and standard optimisation techniques like stochastic gradient descent, the policy and action-value function networks can be trained by iteratively minimising the Bellman residuum,

$$L(\phi, \theta) = \mathbb{E}_{s_t, a_t, r_t} \left[\frac{1}{2} (Q_{\theta}(s_t, a_t) - \hat{Q}_{\theta}(s_t, a_t))^2 \right], \quad (27)$$

$$\text{with } \hat{Q}_{\theta}(s_t, a_t) = r_t(a_t, s_t) + \beta \mathbb{E}_{\mathcal{P}} [Q_{\theta}(s_{t+1}, \mathcal{P}_{\phi}(s_{t+1}))]. \quad (28)$$

We use Haarnoja et al. (2018). The code we used for optimisation is available at <https://github.com/pranz24/pytorch-soft-actor-critic>.

Household learning protocol

Algorithm 1 Training and testing protocol of household agent

Initialise: Environment \mathcal{E} (parameterised model), agent (parameterised by \mathcal{P}_ϕ , Q_θ)

```
for steps = 1 to  $N_{train}$  do
  initialise training episode with random state  $s_t$ 
  while training episode is not done do
    if steps  $\leq N_{burn}$  then
      Take allowed random action  $a_t$ 
    else
      Draw exploration action  $a_t = \mathcal{P}_\phi^{exp}(s_t)$ 
    end if
    Environment returns  $(r_t, s_{t+1}) = \mathcal{E}(s_t, a_t)$ 
    Add transition  $(s_t, a_t, r_t, s_{t+1})$  to memory
    Update  $\mathcal{P}_\phi$ ,  $Q_\theta$  using batch gradient descent from memory
    if  $mod(steps, N_{interval}) = 0$  then
      for test episode = 1 to  $N_{test}$  do
        Record state transitions (*)
      end for
      Save current agent  $(\mathcal{P}_\phi^{steps}, Q_\theta^{steps})$ 
    end if
    State update  $s_t \leftarrow s_{t+1}$ 
    Test episode termination criteria  $(N_{epi}^{max}, d_u^{min})$ 
  end while
end for
Save final agent  $(\mathcal{P}_\phi^{final}, Q_\theta^{final})$ 
```

Model dynamic properties I

The deterministic steady states in the absence of random shocks is characterised by the following set of equations:

$$\text{Euler / Fisher Equation: } R = \frac{\pi}{\beta} \quad (29)$$

$$\text{Money Demand: } m = y \left(\frac{\pi - \beta}{\chi \pi} \right)^{-1/\sigma} \quad (30)$$

$$\text{Monetary Policy: } R = 1 + (R^* - 1) \left(\frac{\pi}{\pi^*} \right)^{\frac{AR^*}{R^* - 1}} \quad (31)$$

$$\text{Fiscal Policy \& GBC: } b = \left(\frac{1}{\beta} - 1 - \gamma \right)^{-1} \left[\gamma_0 + \left(1 - \frac{1}{\pi} \right) m \right] \quad (32)$$

$$\text{Output: } y^{\sigma+\varphi} = 1 \quad (33)$$

Equation (29) and (31) together determine the steady state of inflation:

$$\frac{\pi}{\beta} = 1 + (R^* - 1) \left(\frac{\pi}{\pi^*} \right)^{\frac{AR^*}{R^* - 1}} \quad (34)$$

Model dynamic properties II

In the neighbourhood of either steady state, our model can be described by a linear approximation for π_t and b_t of the form

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{b}_t \end{bmatrix} = \mathbf{B} \begin{bmatrix} \hat{E}_t \pi_{t+1} \\ \hat{E}_t b_{t+1} \end{bmatrix} + \mathbf{C} \begin{bmatrix} \hat{\varepsilon}_t^R \\ \hat{\varepsilon}_t^\tau \\ \hat{\varepsilon}_t^y \end{bmatrix}. \quad (35)$$

Proposition:(Evans and Honkapohja, 2007)] In the linear system given by (35),

- (i) If fiscal policy is passive, $|\gamma - \beta^{-1}| < 1$, the steady state π^* is locally determinate and the steady state π_L is locally indeterminate.
- (ii) If fiscal policy is active, $|\gamma - \beta^{-1}| > 1$, the steady state π^* is locally explosive and the steady state π_L is locally determinate.

Model parameters

parameter	value	description
β	0.9900	discount factor
σ	3.0000	inverse of intertemporal elasticity of consumption and money holdings
φ	1.0000	inverse of Frisch elasticity of labor supply
χ	0.1000	relative preference weight of money holdings
γ_P	0.0200	passive fiscal policy (PFP) coefficient
γ_A	0.0000	active fiscal policy (AFP) coefficient
A	1.3000	Taylor rule coefficient
π^*	1.0100	target gross high-inflation rate (4% net per annum)
π_L	1.0014	implied gross low-inflation steady state (see Figure ??)
ϵ_t^τ	0.0005	monetary policy shock (std. dev.)
ϵ_t^R	0.0005	fiscal policy shock (std. dev.)
ϵ_t^y	0.0005	technology shock (std. dev.)

Baseline model parameterisation. The shock series ϵ_t^τ , ϵ_t^R , ϵ_t^y follow log-normal, normal and normal distributions, with means of one, zero and one, respectively.

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