Fear of Secular Stagnation and the Natural Interest Rate

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DRAFT - PRELIMINARY AND INCOMPLETE

1 Introduction

In the aftermath of the Great Recession the risk-free interest rates sharply dropped. This fall adds up to a secular downward trend in the interest rates that begun in the '80s. While the latter has been attributed to slow moving trends for example in demographics or inequality, the same factors hardly explain this sudden drop in the interest rates. Instead, plausible causes of this sharp decline can be a decrease in productivity that occurred during the crisis or a change in the agents' beliefs.

The aim of this paper is to study the role of agent's beliefs and pessimism in explaining the drop in interest rates during the Great Recession. In particular, we consider that the agents are uncertain about the nature of the shocks that hit the economy: was the decline in GDP persistent but temporary, or permanent? This question has been extensively discussed and the hypothesis of "secular stagnation" arose in the economic debate (Summers, 2014; Gordon, 2012). Our conjecture is that the attribution of a positive probability to the scenario of secular stagnation acts per se as a force that induces a more cautious behavior, that is to consume less and save more, with the consequence of lowering the natural interest rate. The aim of this paper is to verify if this conjecture is empirically relevant, and to quantify the role of beliefs and pessimism in explaining the decline of the interest rates. At the current stage of the project we developed an empirical strategy that serves our purpose: this is what we describe below.

First, we assume that the agents in the economy have limited information on the evolution of productivity. The dynamics of productivity are described by the sum of two

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components, a long run trend and a short run process that captures temporary fluctuations around the trend. The agents can observe productivity but they cannot distinguish its determinants that are latent processes hit by stochastic disturbances. Shocks to productivity, which are also unobservables, can be of two kinds: permanent or transitory.

This hypothesis introduces an extra layer of uncertainty: the agents do not know the probability distribution of future productivity because it depends on the unobserved state of the economy. They consider, then, a set of distributions on future technology. We refer to this situation as *ambiguity*. To understand how ambiguity can affect the decision making process, consider, for example, the case of a temporary shock that makes the realization of productivity different from what the agents expected. Given their information, the agents can also attribute this to a permanent shock or to a wrong assessment on the latent components of productivity. All these three possibilities (and not just that of a temporary shock) are weighted in the consumption/saving decisions.

Ambiguity relaxes the hypothesis that the probability distribution of the future state of the economy is known, as it is often assumed in macroeconomic models. It can be a characteristic of many aspect of the economy. For example Tristani (2009) assumes ambiguity on both future productivity and monetary policy, showing that the latter can affect the natural interest rate; Masolo and Monti (2017) also consider ambiguity on monetary policy, showing the effects on the long run component of inflation; Ilut et al. (2016) study how firms' ambiguity on their competitive environment affect their pricing decisions. In this paper, as in Ilut and Schneider (2014), we assume ambiguity only on productivity: we think this is the relevant channel for the mechanism we want to study. Consider again the example of a temporary shock to technology: since the agents do not distinguish the nature of the shock, they will revise their expected permanent income which is the relevant quantity for their consumption/saving decisions. Blanchard et al. (2017), who also put forward the arguments that a lower assessment on permanent income can explain the weaker demand after the Great recession, focus on revisions in long run forecasts of potential.

We want to verify how much the spectrum of a reduction in permanent income can be responsible for the persistently low interest rates after the crisis. In this mechanism the role of pessimism is key: the higher the agents' pessimism, the greater the probability they attribute to the worst scenario of a permanent reduction in their future income. In order to model pessimism we assume that there is a representative agent endowed with *recursive smooth ambiguity preferences* (Klibanoff et al., 2005, 2009). If the agent is *averse* toward ambiguity she dislikes this additional source of uncertainty and consequently, when she takes decisions, she acts as if the probability distribution on future productivity is biased towards lower values. In this way she takes precautionary decisions against worse outcomes. We refer to this bias as pessimism.[1]

[1]The smooth ambiguity preferences allow us to model also the opposite case of optimism, that materi-

One of the main reasons why we chose the smooth ambiguity approach to model preferences is the flexibility it offers in treating pessimism. It allows to distinguish ambiguity (a characteristic of the decision maker's subjective beliefs) and ambiguity attitude (a characteristic of the decision maker's tastes). An increase in pessimism can be due to an increase in ambiguity aversion or, for given ambiguity aversion, to an increase in ambiguity. In order to model a potential time variation in pessimism we will treat both ambiguity and ambiguity aversion as time varying.

Our main question is empirical, and in order to assess the role of beliefs and pessimism in the aftermath of the Great Recession we need to confront the model with the data. Estimating the complete non-linear model would require a computational effort that is above our possibilities. Then, we obtain an approximation of the solution designing a perturbation technique for models with smooth ambiguity preferences, following the approach of Borovička and Hansen (2014a), Borovička and Hansen (2014b) and Bhandari et al. (2017). This approximation technique, based on series expansions, allows to keep some non-linearities induced by ambiguity-aversion. With respect to Borovička and Hansen (2014a), Borovička and Hansen (2014b) and Bhandari et al. (2017) we face an additional challenge: we also need to approximate the evolution of the agent's beliefs on the latent processes.

We show that a first order approximation can capture the effects of an increase in pessimism, however it cannot disentangle the different contributions of the two sources of pessimism we described above. Then, we develop the approximation up to the second order to capture the additional precautionary motives related to an increase in ambiguity and to distinguish it from an increase in ambiguity-aversion. Finally, we describe an econometric strategy based on particle filtering to approximate the likelihood of non-linear models of the kind presented in this paper.

In the rest of this draft we describe all the steps listed above through a simple example that highlights the core mechanism.

2 Environment

2.1 The process for technology

We assume an exogenous technology process described by the following Dynamic Linear Model:

alizes when the agent loves ambiguity. In the analysis we allow for both possibilities.

$$
\begin{aligned}
\ln(A_t) &= l_t + f_t \qquad (1)\\
l_t &= l_{t-1} + \gamma_t \\
\gamma_t &= (1-\rho_\gamma)\,\bar{\gamma} + \rho_\gamma \gamma_{t-1} + \sigma_\gamma \epsilon_{\gamma t} \\
f_t &= \rho_f f_{t-1} + \sigma_f \epsilon_{ft}
\end{aligned}
$$

where the technology A_t is decomposed into a trend component l_t and a stationary component f_t. We assume that $|\rho_\gamma| < 1$, $|\rho_f| < 1$, and the i.i.d. shocks $(\epsilon_{\gamma t}, \epsilon_{ft})' \sim N(0, I)$. The logarithm of technology can be hit by two types of shocks: $\epsilon_{\gamma t}$ impacts the growth rate of the trend component γ_t and it has a permanent effect to the level; ϵ_{ft} has a persistent but transitory effect, making technology to diverge only temporarily from its trend. The agents observe $\ln(A_t)$ but they do not observe neither its components nor the realization of $\epsilon_{\gamma t}$ and ϵ_{ft}. Parameters are known.

This assumption introduces an additional source of uncertainty. The distribution of technology tomorrow (and its growth rate) given information today is not known because its expected value depends on unobserved components. The agents take into account this additional source of uncertainty when they take decisions and in particular they consider a set of distributions on the future level of technology: the agents face ambiguity.

2.2 The preferences

We assume that a representative agent is endowed with recursive smooth ambiguity preferences (Klibanoff et al., 2005, 2009). Let's consider a simple example that describes the core mechanism: at each time t the representative agent receives A_t as endowment and chooses how much to consume and how much to invest in a risk-free bond. Her budget constraint is the following:

$$C_t + B_{t+1} = A_t + R_t B_t$$

where C_t denotes consumption and B_t is the risk-free bond in zero net supply associated with the interest rate R_t. The value function recursion is defined as follows:

$$V_{s^t}(B_t, \mu_t) = \max_{\{C_t, B_{t+1}\}} u(C_t) + \beta \phi^{-1}\left(E_{\mu_t} \phi\left(E_{\pi_{\theta_t}} V_{(s^t, A_{t+1})}(B_{t+1}, \mu_{t+1})\right)\right)$$

where $s^t = \{A_0, ..., A_t\}$ is the history of observations up to time t and θ_t is the vector of unobserved components in system(1). π_{θ_t} denotes the distribution of A_{t+1} given s^t and θ_t, and μ_t is the Bayesian posterior of θ_t. Given that system (1) is linear and Gaussian and we assume a Gaussian prior over θ_t, μ_t is a Normal distribution with mean m_t and variance Q_t. The inner expectation $E_{\pi_{\theta_t}}$ is computed taking into account all the possible realizations of shocks given θ_t. However, θ_t is not known and the agent needs to compute a second expectation E_{μ_t} over all the possible values of the latent vector. These two integrals do not reduce thanks to the presence of the increasing function ϕ, that denotes the agent's ambiguity attitude. We assume:

$$\phi(z, \alpha_t) = -\frac{1}{\alpha_t} \exp\{-\alpha_t z\}$$

where α_t is a time-varying coefficient of ambiguity attitude. When $\alpha_t > 0$, the function ϕ is concave and the agent gives higher weight to lower expected continuation values. In this case the agent is defined as ambiguity averse. Conversely, when $\alpha_t < 0$, the agent is defined ambiguity loving. In case of $\alpha_t = 0$, the agent is ambiguity neutral and her choices are observationally equivalent to that of a Bayesian decision maker. Note how the smooth ambiguity preferences allow to distinguish ambiguity and ambiguity-aversion. The former is a characteristic of the decision maker's subjective beliefs, and in this setting higher ambiguity corresponds to higher variance of the posterior distribution on the unobserved vector: we labelled it Q_t. Ambiguity attitude, on the other hand, is a characteristic of the decision maker's tastes and it tells how much the agent dislikes (or like) the uncertainty over θ_t.

In equilibrium, the gross risk-free interest rate is as follows:

$$R_{t+1}^{-1} = E_{\mu_t}\left[\xi_t E_{\pi_{\theta_t}}\left(\beta \frac{A_t}{A_{t+1}}\right)\right] \tag{2}$$

where

$$\xi_t \equiv \frac{\exp\{-\alpha_t E_{\theta_t} V_{t+1}\}}{E_{\mu_t}\left[\exp\{-\alpha_t E_{\theta_t} V_{t+1}\}\right]} \tag{3}$$

With respect to the standard case of complete information, the Euler equation (2) has two differences: first, the growth rate of technology depends on the latent vector θ_t which is unknown. Then, the agent forms her beliefs conditional on the observed technology using the Bayes rule. Second, when taking decisions, she acts as if her posterior distribution is distorted through the presence of ξ_t. The latter is in fact a Radon-Nikodym derivative with respect to the Bayesian posterior distribution:

$$\xi_t = \frac{d\mu_t^*}{d\mu_t}$$

In particular, when the time-varying parameter α_t is positive, the agent gives more weight to lower continuation values, and the distorted distribution has a bias with respect to the Bayesian posterior that we call *pessimism*.

The definition of the distortion shows that there are two sources of pessimism. The first is the time-varying parameter α_t that affects the slope of the negative exponential in (3). Intuitively this time varying coefficient controls the agent's tastes: the higher is α_t, the more the agent is worried about bad outcomes (lower expected continuation values). The second source of pessimism is the agent's uncertainty Q_t, that is the variance of the Bayesian posterior distribution over θ_t. To understand the intuition consider the case of a positive α_t: higher uncertainty implies a bigger probability over bad outcomes, leading to a more cautious behavior.

We allow both these sources of pessimism to vary over time. As in Bhandari et al. (2017), who work with multiplier preferences,[2] we assume that the coefficient of ambiguity-attitude follows an AR(1) process around a mean:

$$\alpha_t = (1 - \rho_\alpha)\bar{\alpha} + \rho_\alpha \alpha_{t-1} + \sigma_\alpha \epsilon_{\alpha t} \ . \tag{4}$$

In this way, agent's tastes can change over time, allowing for periods of higher or lower pessimism, and eventually also for periods of optimism (when α_t becomes negative).

In addition, we also introduce time-varying uncertainty: in each period the variance of the prior distribution over θ_t is hit by a shock that can reduce or increase the uncertainty perceived by the agent. Formally, at period $t-1$ the posterior distribution of θ_{t-1} is a Normal $N(m_{t-1}, Q_{t-1})$. In a standard filtering process this distribution becomes the prior that the agent will use to update her beliefs over θ_t. We assume, instead, that at each time t, before the agent observes the outcome A_t, the variance of her prior is:

$$Q^*_{t-1} = Q_{t-1} e^{\sigma_\eta \eta_t} \tag{5}$$

where $\eta_t \sim N(0,1)$. Without the shock η_t the variance Q_t converges to the time invariant variance of the steady state Kalman filter Q. The presence of the shock introduces variations around Q, so that when the agent takes her decisions she may feel more or less confident about her knowledge over θ.

We could introduce time variation in uncertainty also assuming that system (1) has stochastic volatility or time varying parameters. The assumption of equation (5) is both simple and more in line with the one of equation (4), so that we treat both sources of pessimism in a similar way.

[2]The multiplier preferences are one of the specifications of the robust preference approach by Hansen and Sargent (2011). In the multiplier preferences approach there is a penalty parameter that plays an analogous role to our parameter of ambiguity aversion. However, the penalty parameter controls both ambiguity and ambiguity aversion, while in our case we are able to distinguish these two features.

3 Approximation of a general equilibrium model with smooth ambiguity preferences

Computing the solution of a model under the assumption of recursive smooth ambiguity preferences in not straightforward because it entails the computation of the expected value function that enters in the definition of ξ_t. The use of numerical methods, as in Collard et al. (2018), is not an option if the goal is estimation.

The alternative that we consider is to approximate the model using a perturbation technique based on series expansion (Holmes, 1998; Lombardo, 2010). The risk of the approximation is that we loose the non-linear effects we are interested in: a linear approximation, for example, would cancel all the effects of ambiguity aversion, as clear from equations (2) and (3). Borovička and Hansen (2014a), Borovička and Hansen (2014b) face the same issue working with multiplier preferences and propose to perturb jointly the standard deviations and the parameter of ambiguity aversion: in this way the distortion ξ_t is approximated up to one order higher with respect to the rest of the model. We apply this idea to models with smooth ambiguity preferences: with respect to the multiplier preference specification, we have a learning process over the vector of latent variables. Then, we have the additional challenge to keep track of the evolution of beliefs.

3.1 The perturbation technique

3.1.1 The evolution of beliefs

Let's assume that y_t is a variable that depends on a set of unobservable components collected in the $(v \times 1)$ vector θ_t, that follows a Dynamic Linear model (DLM):

$$\begin{aligned} y_t &= F\theta_t + \Omega\omega_t^y \\ \theta_t &= G_0\bar{\theta} + G_1\theta_{t-1} + \Sigma\omega_t^\theta \end{aligned} \tag{6}$$

where F is $(1 \times v)$, G_0 and G_1 are $(v \times v)$, Ω is $(1 \times k_3)$ and Σ is $(v \times k_2)$; $\omega_t^y \sim N_{k_3}(0, I)$ and $\omega_t^\theta \sim N_{k_2}(0, I)$. Let's assume that G_1 is a stationary matrix.[3] Note that under this assumption if $G_0 = (I - G_1)$, $\bar{\theta}$ is the steady-state of θ_t, and we assume it is known.

We assume that in each period the agents observe the realization of y_t, but they never observe the realization of the shocks $\omega_t^y, \omega_t^\theta$ and of the latent vector θ_t. All the parameters are known.

The agents form subjective beliefs about the latent vector θ_t and update them in each period given the observation y_t applying the Kalman filter. Since we want to allow

[3] In system (1) this assumption does not hold: we will proceed considering as observable the growth rate of technology.

for time-variation in uncertainty, we assume that in each period the variance-covariance matrix of the prior distribution can be hit by an iid shock η_t, as in equation (5). The timing is as follows: at time $t-1$ the beliefs of the agent are summarized by their posterior distribution:

$$\theta_{t-1}|y_{t-1} \sim N(m_{t-1}, Q_{t-1}) \ .$$

The variance Q_{t-1} is the relevant matrix for the decisions that the agent takes at $t-1$. However, before observing the realization of y_t and updating his beliefs over the latent vector, the variance Q_{t-1} is hit by the shock η_t as in equation (5). Then the new data arrive and the agent updates her beliefs using the Bayes rule and deriving the posterior distribution:

$$\theta_t|y_t \sim N(m_t, Q_t) \ .$$

It is convenient to rewrite the posterior mean and variance in recursive form:

$$m_t = (I - K_tF)\, G_0\bar{\theta} + (I - K_tF)\, G_1 m_{t-1} + K_tF\theta_t + K_t\Omega\omega_t^y \tag{7}$$

and

$$Q_t = (I - K_tF)\, G_1 Q_{t-1}^{*} G_1' + (I - K_tF)\, W \tag{8}$$

where K_t is the Kalman Gain. Note that the DLM specified in equation (6) is time-invariant, meaning that $F, G_0, G_1, \Sigma, \Omega$ are constant. In the absence of uncertainty shocks (i.e. when $\eta_t = 0$), the filter converges to a steady state solution, with time-invariant posterior variance Q and constant Kalman gain K.

3.1.2 The equilibrium conditions

Let x_t be a $(n \times 1)$ vector that can include both control and state variables (endogenous and exogenous), except for the vector of the latent exogenous state variables θ_t. Recall that the scalar α_t is the coefficient of ambiguity aversion for which we assume the exogenous AR(1) process (4).

The system of equilibrium conditions of our model with smooth ambiguity is as follows:

$$0 = E_{\mu_t}\left[\xi_t E_{\theta_t}\left[g^1\left(x_{t+1}, x_t, x_{t-1}, \theta_{t+1}, \theta_t, \omega_{t+1}^x, \omega_t^x, \omega_{t+1}^\theta, \omega_{t+1}^y\right)\right]\right] \tag{9}$$

$$0 = E_{\theta_t}\left[g^2\left(x_{t+1}, x_t, x_{t-1}, \theta_{t+1}, \theta_t, \omega_{t+1}^x, \omega_t^x, \omega_{t+1}^\theta, \omega_{t+1}^y\right)\right] \tag{10}$$

where g_t^1 is an $(n_1 \times 1)$ vector function that collects all the equilibrium conditions in which it appears the distorted expectation of θ_t, g_t^2 is an $(n_2 \times 1)$ vector function that collects all the equilibrium conditions in which θ_t appears without expectation. The sum of the equilibrium conditions is n: $n_1 + n_2 = n$. Finally, ω_{t+1}^x is a $(k_1 \times 1)$ vector of i.i.d.Gaussian innovations: $\omega^x_{t+1} \sim N(0, I)$. To define the equilibrium we need to take into account the dynamics of θ_t in the state equation of system (6), the dynamics of α_t in equation (4), the evolution of the beliefs m_t in equation (7) and the evolution of the variance of the posterior Q_t in equation (8).

3.1.3 The approximated dynamics

The recursive solution of the model is defined by the endogenous law of motion of x_t that satisfies the equilibrium conditions of the model. We define the endogenous law of motion as[4]

$$x_{t+1} = \psi(x_t, m_{t+1}, Q_{t+1}, \alpha_{t+1}, \theta_{t+1}, \omega^x_{t+1}) \; .$$

The endogenous law of motion ψ is unknown and needs to be solved for from the set of equilibrium conditions. In order to approximate it let's consider the following class of models indexed by the perturbation parameter q, that scales the volatility of the shocks:

$$x_{t+1}(q) = \psi\left(x_t(q), m_{t+1}(q), Q_{t+1}(q), \alpha_{t+1}(q), \theta_{t+1}(q), q\omega^x_{t+1}, q\right)$$

where

$$\begin{aligned}
m_{t+1}(q) &= \left(I - K_{t+1}(q)F\right) G_0\bar{\theta} + \left(I - K_{t+1}(q)F\right) G_1 m_t(q) + K_{t+1}(q)F\theta_{t+1}(q) + K_{t+1}(q)q\Omega\omega^y_{t+1} \\
Q_{t+1}(q) &= \left(I - K_{t+1}(q)F\right)\left(G_1 Q^*_t(q) G'_1 + W\right) \\
\alpha_{t+1}(q) &= (1-\rho_\alpha)\bar{\alpha} + \rho_\alpha \alpha_t(q) + q\sigma_\alpha \epsilon_{\alpha,t+1} \\
\theta_{t+1}(q) &= G_0\bar{\theta} + G_1\theta_t(q) + q\Sigma\omega^\theta_{t+1}
\end{aligned}$$

with

$$\begin{aligned}
K_{t+1}(q) &= H_{t+1}(q)F'\left[FH_{t+1}(q)F' + V\right]^{-1} \\
H_{t+1}(q) &= G_1 Q^*_t(q) G'_1 + W \\
Q^*_t(q) &= Q_t(q)e^{q\sigma_\eta \eta_{t+1}}
\end{aligned}$$

Assume there exists a series expansion of $x_{t+1}(q), m_{t+1}(q), Q_{t+1}(q), \alpha_{t+1}(q), \theta_{t+1}(q)$ around $q = 0$:

$$\begin{aligned}
x_{t+1}(q) &\approx \bar{x} + qx_{1t+1} + \frac{q^2}{2}x_{2t+1} \\
m_{t+1}(q) &\approx \bar{m} + qm_{1t+1} + \frac{q^2}{2}m_{2t+1}
\end{aligned}$$

[4] In general, the endogenous law of motion can be a function of θ_t, if the latter is known by some agents, or if it enters in market clearing conditions or in the budget constraints. We assume that nobody knows the true value of θ_t.

$$
\begin{aligned}
Q_{t+1}(q) &\approx Q + qQ_{1t+1} + \frac{q^2}{2}Q_{2t+1} \\
\alpha_{t+1}(q) &\approx \bar{\alpha} + q\alpha_{1t+1} \\
\theta_{t+1}(q) &\approx \bar{\theta} + q\theta_{1t+1}
\end{aligned}
$$

where the expansion of α_{t+1} and θ_{t+1} are at the first-order since they are linear. The zeroth-order approximation is [5]

$$
\begin{aligned}
\bar{x} &= \psi\left(\bar{x}, \bar{m}, Q, \bar{\alpha}, \bar{\theta}, 0, 0\right) \\
\bar{m} &= \bar{\theta} \\
Q &= (I - KF)\,H
\end{aligned}
$$

The approximation for x_t is defined by

$$
x_{1t+1} = \psi_x x_{1t} + \psi_m m_{1t+1} + \psi_Q vec(Q_{1t+1}) + \psi_\alpha \alpha_{1t+1} + \psi_\theta \theta_{1t+1} + \psi_{\omega_x}\omega^x_{t+1} + \psi_q \tag{11}
$$

where $\underbrace{\psi_Q}_{(n\times v^2)} = \frac{\partial \psi}{\partial vec(Q_{t+1})}$ and $\underbrace{vec(Q_{1t+1})}_{(v^2\times 1)} = vec\frac{\partial Q_{t+1}(q)}{\partial q}$, and

$$
\begin{aligned}
x_{2t+1} = \psi_x x_{2t} &+ \psi_{xx}\left(x_{1t}\otimes x_{1t}\right) + 2\psi_{xm}\left(m_{1t+1}\otimes x_{1t}\right) + 2\psi_{xQ}\left(vecQ_{1t+1}\otimes x_{1t}\right) + \\
&+ 2\psi_{x\alpha}\alpha_{1t+1}x_{1t} + 2\psi_{x\theta}\left(\theta_{1t+1}\otimes x_{1t}\right) + 2\psi_{x\omega_x}\left(\omega^x_{t+1}\otimes x_{1t}\right) + 2\psi_{xq}x_{1t} + \psi_m m_{2t+1} + \\
&+ \psi_{mm}\left(m_{1t+1}\otimes m_{1t+1}\right) + 2\psi_{mQ}\left(vecQ_{1t+1}\otimes m_{1t+1}\right) + 2\psi_{m\alpha}\alpha_{1t+1}m_{1t+1} + \\
&+ 2\psi_{m\theta}\left(\theta_{1t+1}\otimes m_{1t+1}\right) + 2\psi_{m\omega_x}\left(\omega^x_{t+1}\otimes m_{1t+1}\right) + 2\psi_{mq}m_{1t+1} + \psi_Q vecQ_{2t+1} + \\
&+ \psi_{QQ}\left(vecQ_{1t+1}\otimes vecQ_{1t+1}\right) + 2\psi_{Q\alpha}\alpha_{1t+1}vecQ_{1t+1} + 2\psi_{Q\theta}\left(\theta_{1t+1}\otimes vecQ_{1t+1}\right) + \\
&+ 2\psi_{Q\omega_x}\left(\omega^x_{t+1}\otimes vecQ_{1t+1}\right) + 2\psi_{Qq}vecQ_{1t+1} + \psi_{\alpha\alpha}\alpha_{1t+1}\alpha_{1t+1} + 2\psi_{\alpha\theta}\theta_{1t+1}\alpha_{1t+1} + \\
&+ 2\psi_{\alpha\omega_x}\omega^x_{t+1}\alpha_{1t+1} + 2\psi_{\alpha q}\alpha_{1t+1} + \psi_{\theta\theta}\left(\theta_{1t+1}\otimes\theta_{1t+1}\right) + 2\psi_{\theta\omega_x}\left(\omega^x_{t+1}\otimes\theta_{1t+1}\right) + 2\psi_{\theta q}\theta_{1t+1} + \\
&+ \psi_{\omega_x\omega_x}\left(\omega^x_{t+1}\otimes\omega^x_{t+1}\right) + 2\psi_{\omega_x q}\omega^x_{t+1} + \psi_{qq}
\end{aligned} \tag{12}
$$

The first and second order terms for approximation of the variance-covariance matrix are:

$$
Q_{1t+1} = ZQ_{1t}Z' + ZQZ'\sigma_\eta\eta_{t+1} \tag{13}
$$

and

$$
Q_{2t+1} = ZQ_{2t}Z' + 2\sigma_\eta\eta_{t+1}ZQ_{1t}Z' + \sigma^2_\eta\eta^2_{t+1}ZQZ' - 2ZQ_{1t}DQ_{1t}Z' - 2\sigma_\eta\eta_{t+1}ZQ_{1t}DQZ' +
$$

[5] Note that $\bar{m}$ denotes the beliefs (the mean of the posterior distribution of θ) in the deterministic steady-state (i.e. when all the shocks are zero). It is different from the beliefs given by the steady-state solution of the Kalman filter (when just the uncertainty shock is zero).

$$- 2\sigma_\eta \eta_{t+1} ZQDQ_{1t}Z' - 2\sigma_\eta^2 \eta_{t+1}^2 ZQDQZ' \tag{14}$$

where $Z \equiv (I - KF) G_1$, K is the steady state Kalman gain and $D \equiv G_1' F' [FHF' + V]^{-1} FG_1$.

For the mean of the posterior distribution we have

$$m_{1t+1} = (I - KF) G_1 m_{1t} + KF\theta_{1t+1} + K\Omega\omega_{t+1}^y \tag{15}$$

where

$$\theta_{1t+1} = G_1 \theta_{1t} + \Sigma\omega_{t+1}^\theta$$

is the first order term in the approximation of θ_t, and

$$\begin{aligned} m_{2t+1} = Zm_{2t} - 2\tilde{S}\left(vec(Q_{1t}) \otimes m_{1t}\right) + 2\tilde{S}\left(vec(Q_{1t}) \otimes \theta_{1t}\right) - 2\sigma_\eta \eta_{t+1} ZQDm_{1t} + 2\sigma_\eta \eta_{t+1} ZQD\theta_{1t} + \\ + 2Z\left[Q_{1t} + Q\sigma_\eta \eta_{t+1}\right] G_1' F' \left[FHF' + V\right]^{-1} \left(F\Sigma\omega_{t+1}^\theta + \Omega\omega_{t+1}^y\right) \end{aligned} \tag{16}$$

where

$$\tilde{S} \equiv \left[(vecD)' \otimes Z_{\bullet j}\right]_{j=1}^v$$

The first-order term in the approximation for α_{t+1} is:

$$\alpha_{1t+1} = \rho_\alpha \alpha_{1t} + \sigma_\alpha \epsilon_{\alpha,t+1} \tag{17}$$

The unknowns of the system we need to find in order to obtain the approximated solution of the model are ψ_x, ψ_m, ψ_Q, ψ_α, ψ_θ, ψ_{ω_x} and ψ_q; ψ_{xx}, ψ_{xm}, ψ_{xQ}, $\psi_{x\alpha}$, $\psi_{x\theta}$, $\psi_{x\omega_x}$, ψ_{xq}, ψ_{mm}, ψ_{mQ}, $\psi_{m\alpha}$, $\psi_{m\theta}$, $\psi_{m\omega_x}$, ψ_{mq}, ψ_{QQ}, $\psi_{Q\alpha}$, $\psi_{Q\theta}$, $\psi_{Q\omega_x}$, ψ_{Qq}, $\psi_{\alpha\alpha}$, $\psi_{\alpha\theta}$, $\psi_{\alpha\omega_x}$, $\psi_{\alpha q}$, $\psi_{\theta\theta}$, $\psi_{\theta\omega_x}$, $\psi_{\theta q}$, $\psi_{\omega_x\omega_x}$, $\psi_{\omega_x q}$, ψ_{qq}. In order to find those coefficients, we need to take the zero-th, first-order and second-order expansions of the system of equilibrium conditions (9),(10):

$$\begin{aligned} 0 &= g_{0t}^1 \\ 0 &= g_{0t}^2 \\ \\ 0 &= E_{\mu_t}\left[\xi_{0t} E_{\theta_t}\left(g_{1t}^1\right)\right] \\ 0 &= E_{\theta_t}\left(g_{1t}^2\right) \\ \\ 0 &= E_{\mu_t}\left[\xi_{0t} E_{\theta_t}\left(g_{2t}^1\right)\right] + 2E_{\mu_t}\left[\xi_{1t} E_{\theta_t}\left(g_{1t}^1\right)\right] \end{aligned} \tag{18}$$

$$0 = E_{\theta_t}\left(g_{2t}^2\right)$$

where ξ_{0t} and g_{0t}^i are respectively $\xi_t(q)$ and $g_t^i(q)$ evaluated for $q = 0$, ξ_{1t} and g_{1t}^i are respectively their first derivatives with respect to q , evaluated at $q = 0$, and ξ_{2t} and g_{2t}^i are respectively their second derivatives with respect to q , evaluated at $q = 0$, $i = 1, 2$.

To approximate the beliefs distortion ξ_t we first need to approximate the continuation value.

3.1.4 Expansion of the continuation value recursion

The continuation value recursion when agents have smooth ambiguity preferences is as follows:

$$V_t = u(C_t) + \beta\phi^{-1}\left[E_{\mu_t}\phi\left(E_{\theta_t}V_{t+1}\right)\right] \tag{19}$$

where C_t denotes consumption, and we assume that

$$\phi(y) = -\frac{1}{\alpha_t}\exp\{-\alpha_t y\} \tag{20}$$

Let's assume that $u(C_t) = \ln(C_t)$. Since in equilibrium C_t has a unit root, the utility function u_t has a unit root as well, implying that also V_t is non-stationary. In order to proceed with the approximation of the value function recursion, we want to decompose it in a stationary component and its trend component. We start decomposing in this way the utility function.

$$\begin{aligned} u\left(C_t\right) &= \ln C_t \\ &= \hat{u}\left(\hat{C}_t\right) + \ln A_t \end{aligned} \tag{21}$$

where $\hat{C}_t = \frac{C_t}{A_t}$ and $\ln(A_t)$ is defined by system (1). Define $\varphi\left(\theta_t, \omega_{t+1}^\theta, \omega_{t+1}^y\right)$ as:

$$\begin{aligned} \ln\left(\frac{A_{t+1}}{A_t}\right) &= \varphi\left(\theta_t, \omega_{t+1}^\theta, \omega_{t+1}^y\right) \\ &= F\left(G_0\bar{\theta} + G_1\theta_t + \Sigma\omega_{t+1}^\theta\right) + \Omega\omega_{t+1}^y \end{aligned} \tag{22}$$

Then, we can write the value function as the sum of a stationary component and a non-stationary component, as follows

$$V_t = \hat{V}_t + (1-\beta)^{-1}\ln\left(A_t\right) \tag{23}$$

and the recursion of the stationary component of the value function is:

$$\hat{V}_t = \hat{u}(x_t) - \frac{\beta}{\alpha_t} \ln E_{\mu_t} \left[\exp \left(-\alpha_t \left(E_{\theta_t} \hat{V}_{t+1} + (1-\beta)^{-1} E_{\theta_t} \varphi \left(\theta_t, \omega^{\theta}_{t+1}, \omega^{y}_{t+1} \right) \right) \right) \right] \tag{24}$$

For the approximation of the stationary value function we follow Borovička and Hansen (2014a), Borovička and Hansen (2014b) and Bhandari et al. (2017): we perturb jointly the volatility of the shocks and the coefficient of ambiguity aversion with the parameter q.

$$\hat{V}_t(q) = \hat{u}(x_t(q), q) - \beta \frac{q}{(\bar{\alpha} + \alpha_{1t})} \times$$
$$\times \ln E_{\mu_t} \left[\exp \left(-\frac{(\bar{\alpha} + \alpha_{1t})}{q} \left(E_{\theta_t} \hat{V}_{t+1}(q) + (1-\beta)^{-1} E_{\theta_t} \varphi \left(\theta_t(q), q\omega^{\theta}_{t+1}, q\omega^{y}_{t+1}, q \right) \right) \right) \right] \tag{25}$$

In the last equation we need to consider a series expansions for $\hat{u}(x_t)$ and $\varphi(\theta_t, \omega^{\theta}_{t+1}, \omega^{y}_{t+1})$.

The stationary value function recursion is approximated as:

$$\hat{V}_t(q) \approx \bar{V} + q\hat{V}_{1t} + \frac{q^2}{2} \hat{V}_{2t} \tag{26}$$

The zeroth-order approximation of the value function, $\bar{V}$, is obtained evaluating the perturbed value function recursion at $q = 0$:

$$\bar{V} = (1-\beta)^{-1} \left[\bar{u} + \frac{\beta}{1-\beta} \bar{\varphi} \right] \tag{27}$$

where $\bar{\varphi} \equiv \varphi(\bar{\theta}, 0, 0, 0)$.

To obtain the first-order approximation of the value function recursion, $\hat{V}_{1t}$, let's take the derivative of equation (25) with respect to q, and evaluate it at $q = 0$:

$$\hat{V}_{1t} = \hat{u}_{1t} - \frac{\beta}{(\bar{\alpha} + \alpha_{1t})} \ln E_{\mu_t} \left[\exp \left(- (\bar{\alpha} + \alpha_{1t}) E_{\theta_t} \left(\hat{V}_{1t+1} + \frac{\varphi_{1t+1}}{1-\beta} \right) \right) \right] \tag{28}$$

Taking the second-order derivative of equation (25) with respect to q, evaluated for $q = 0$, we obtain the second-order approximation of the value function recursion:

$$\hat{V}_{2t} = \hat{u}_{2t} + \beta E_{\mu_t} \left[\frac{\exp \left[- (\bar{\alpha} + \alpha_{1t}) E_{\theta_t} (\hat{V}_{1t+1} + \frac{\varphi_{1t+1}}{1-\beta}) \right]}{E_{\mu_t} \left[\exp \left[- (\bar{\alpha} + \alpha_{1t}) E_{\theta_t} (\hat{V}_{1t+1} + \frac{\varphi_{1t+1}}{1-\beta}) \right] \right]} E_{\theta_t} \left(\hat{V}_{2t+1} \right) \right] \tag{29}$$

We assume that [6]

$$\hat{V}_t = \hat{V}\left(x_t, m_t, Q_t, \alpha_t\right)$$

We can perturb the stationary value function as follows:

$$\hat{V}_t(q) = \hat{V}\left(x_t(q), m_t(q), Q_t(q), \alpha_t(q), q\right)$$

Then, the first and second order terms in the approximated value function are:

$$\hat{V}_{1t} = V_x x_{1t} + V_m m_{1t} + V_Q vec(Q_{1t}) + V_\alpha \alpha_{1t} + V_q \tag{30}$$

$$\begin{aligned}\hat{V}_{2t} = {} & V_x x_{2t} + V_{xx}\left(x_{1t}\otimes x_{1t}\right) + 2V_{xm}\left(m_{1t}\otimes x_{1t}\right) + 2V_{xQ}\left(vec(Q_{1t})\otimes x_{1t}\right) + 2V_{x\alpha}\alpha_{1t}x_{1t}+ \\ & + 2V_{xq}x_{1t} + V_m m_{2t} + V_{mm}\left(m_{1t}\otimes m_{1t}\right) + 2V_{mQ}\left(vec(Q_{1t})\otimes m_{1t}\right) + 2V_{m\alpha}\alpha_{1t}m_{1t}+ \\ & + 2V_{mq}m_{1t} + V_Q vec(Q_{2t}) + V_{QQ}\left(vec(Q_{1t})\otimes vec(Q_{1t})\right) + 2V_{Q\alpha}\alpha_{1t}vec(Q_{1t})+ \\ & + 2V_{Qq}vec(Q_{1t}) + V_{\alpha\alpha}\left(\alpha_{1t}\otimes\alpha_{1t}\right) + 2V_{\alpha q}\alpha_{1t} + V_{qq}\end{aligned} \tag{31}$$

where $\underbrace{V_Q}_{(1\times v^2)} = \frac{\partial \hat{V}_t}{\partial vec(Q_t)}$, $\underbrace{vec(Q_{1t})}_{(v^2\times 1)} = vec\frac{\partial Q_t(q)}{\partial q}$, $\underbrace{V_{QQ}}_{(1\times v^4)} = \frac{\partial^2 \hat{V}_t}{\partial vec(Q_t)^2}$ and $\underbrace{vec(Q_{2t})}_{(v^4\times 1)} = vec\frac{\partial^2 Q_t(q)}{\partial q^2}$.

Substituting the approximations $\hat{V}_{1t}$ and $\hat{V}_{2t}$ respectively in (28) and (29), we obtain all the coefficients of the approximated value function recursion (given the coefficients of the endogenous law of motion) using the method of undetermined coefficients.

3.1.5 Approximation of belief distortion

We now approximate the belief distortion ξ_t, perturbing jointly the volatility of the shocks and the coefficient of ambiguity aversion, as in the previous section, and using the approximation of the value function. Let's consider the class of models:

$$\xi_t(q) = \frac{\exp\left[-\frac{(\bar{\alpha}+\alpha_{1t})}{q}E_{\theta_t}\left(\hat{V}_{t+1}(q) + (1-\beta)^{-1}\varphi\left(\theta_t(q), q\omega^\theta_{t+1}, q\omega^y_{t+1}, q\right)\right)\right]}{E_{\mu_t}\left[\exp\left[-\frac{(\bar{\alpha}+\alpha_{1t})}{q}E_{\theta_t}\left(\hat{V}_{t+1}(q) + (1-\beta)^{-1}\varphi\left(\theta_t(q), q\omega^\theta_{t+1}, q\omega^y_{t+1}, q\right)\right)\right]\right]} \tag{32}$$

and assume:

$$\xi_t \approx \xi_{0t} + q\xi_{1t} + \frac{q^2}{2}\xi_{2t}\ .$$

[6] Given our assumption that the agents do not observe θ, the stationary value function does not depend on it.

Substituting the approximated value function and simplifying for q, we obtain:

$$\xi_t(q) = \frac{\exp\left[-(\bar{\alpha}+\alpha_{1t})E_{\theta_t}\left(\hat{V}_{1t+1}+\frac{q}{2}\hat{V}_{2t+1}+\frac{q^2}{6}\hat{V}_{3t+1}+(1-\beta)^{-1}\varphi_{1t+1}\right)\right]}{E_{\mu_t}\left[\exp\left[-(\bar{\alpha}+\alpha_{1t})E_{\theta_t}\left(\hat{V}_{1t+1}+\frac{q}{2}\hat{V}_{2t+1}+\frac{q^2}{6}\hat{V}_{3t+1}+(1-\beta)^{-1}\varphi_{1t+1}\right)\right]\right]} \tag{33}$$

Thanks to this perturbation, the zeroth-order approximation of ξ_t is not simply equal to one and we are able to keep some of the effects of ambiguity aversion even when we approximate the solution to the first-order (in fact, as shown in (18), ξ_{0t} enters in the first-order approximation of the equilibrium conditions).

The zeroth-order approximation is:

$$\xi_{0t} = \frac{\exp\left(-\alpha_t B\theta_{1t}\right)}{E_{\mu_t}\left[\exp\left(-\alpha_t B\theta_{1t}\right)\right]}$$

where $B \equiv \left(V_x\psi_m KFG_1 + V_x\psi_\theta G_1 + V_m KFG_1 + (1-\beta)^{-1}\varphi_\theta\right)$.Taking the Bayesian expectation at the denominator we can rewrite ξ_{0t} as:

$$\xi_{0t} = \frac{\exp\left(-\alpha_t B\theta_{1t}\right)}{\exp\left(-\alpha_t B\left(m_t-\bar{m}\right)+\frac{\alpha_t^2 BQ_tB'}{2}\right)} \tag{34}$$

Note that $E_{\mu_t}\xi_{0t} = 1$: ξ_{0t} is a Radon-Nykodim derivative with respect to μ_t, and it changes the Bayesian posterior distribution into a new measure:

$$\xi_{0t} = \frac{d\mu_t^*}{d\mu_t}$$

where μ_t^* is the new probability measure on θ_{1t}.

To find the new distribution of θ_{1t} under the probability measure μ_t* we can multiply the distortion ξ_{0t} times the probability density function of θ_{1t} under the measure μ_t:

$$\xi_{0t}f_{\mu_t}(\theta_{1t}) \propto exp\left\{-\frac{1}{2}\left(\theta_{1t}'-2\left[(m_t-\bar{m})'-\alpha_t BQ_t\right)\right]Q_t^{-1}\theta_{1t}\right\} \tag{35}$$

We recognize that equation (35) shows the kernel of a multivariate Normal distribution with mean $(m_t-\bar{m})-\alpha_t Q_t B'$ and variance Q_t. Then, the distribution of θ_t under the new measure is:

$$\theta_t \sim N\left(m_t-\alpha_t Q_t B', Q_t\right) \tag{36}$$

The effect of ξ_t is to shift the mean of the Bayesian posterior distribution, leaving the variance unchanged: it creates a bias. In particular, when α_t is positive the effect is to shift the distribution toward more negative outcomes, and we call this bias *pessimism*.

The possibility to distinguish the two sources of pessimism is a peculiarity of the smooth ambiguity preferences. However, from (36) we see that α_t and Q_t enter multiplicatively. Then, in order to disentangle the contribution of a shock to ambiguity attitude and a shock to uncertainty, we need to consider that while a shock to α_t only changes the mean of the distorted distribution, a shock to uncertainty changes both the mean and the variance.

4 The core mechanism

4.1 Endowment economy

Consider again our simple endowment economy. We specify the dynamics of the exogenous productivity in terms of its growth rate: the vector of latent variables in (6) is:

$$\theta_t = \begin{bmatrix} \gamma_t \\ f_t \\ f_{t-1} \end{bmatrix} .$$

The first order approximation of the risk-free interest rate is proportional to the expected value of the growth rate of technology. Under complete information the agent knows the value of θ_t, and the risk-free rate is:

$$R_{1t}^{C} = \beta^{-1} e^{\bar{\gamma}} \begin{bmatrix} \rho_\gamma & \rho_f - 1 & 0 \end{bmatrix} \theta_{1t} \tag{37}$$

We assume, instead, that the agent cannot observe the vector θ_t and she consider a Bayesian posterior distribution. Under neutrality toward ambiguity we are in the case of subjective expected utility and the risk-free interest rate is:

$$R_{1t}^{B} = \beta^{-1} e^{\bar{\gamma}} \begin{bmatrix} \rho_\gamma & \rho_f - 1 & 0 \end{bmatrix} m_{1t} \tag{38}$$

Finally, under the smooth ambiguity preferences, the expected value of the growth rate of technology is obtained distorting the Bayesian posterior, and the risk-free rate is:

$$R_{1t} = \beta^{-1} e^{\bar{\gamma}} \begin{bmatrix} \rho_\gamma & \rho_f - 1 & 0 \end{bmatrix} \Bigg[m_{1t} - \underbrace{\left(\bar{\alpha} Q_{1t} + Q \alpha_{1t} + \bar{\alpha} Q\right) B'}_{Pessimism} \Bigg] \tag{39}$$

The equation above shows how an increase in pessimism, given by an increase in α_t or equivalently to an increase in Q_t, has the effect of lowering the risk-free rate. The intuition is straightforward: higher pessimism leads to a more cautious behavior with respect to negative scenarios. The agent acts as if her expectation about the growth rate of technology is lower: she would like to decrease consumption and save in the risk-free asset. In equilibrium, a lower interest rate is required to let her consume the endowment A_t.

To clarify the argument we can simulate the dynamics of this simple economy. Let's calibrate $\bar{\gamma}$ to 0.005, both the autoregressive parameter ρ_γ and ρ_f to 0.99, the subjective discount factor β to 0.995, the variances σ_γ to 0.001 and σ_f to 0.01, and the mean of ambiguity aversion $\bar{\alpha}$ to 0.1. Consider the effect of a negative temporary shock ϵ_{ft}. Figure 1 shows the true dynamics of γ_t and f_t and compares them with the Bayesian and the distorted beliefs. In particular we show the online filtered expectations. When a temporary shock hits technology, the agent observes a lower A_t than what she expected. She attributes part of it to a shift in her estimated value for γ_t: this can result from both a permanent shock or to a wrong assessment in her previous estimation. As time goes on, she learns the truth (there are no other shocks) and her beliefs converge to the true values. When she takes decisions, however, she acts as if her beliefs were distorted (the continuous red line). The pattern is the same as the Bayesian posterior, with a constant wedge.

In Figure 2 we plot the response of the risk-free rate for the three benchmarks. Under complete information the agent knows that the shock is temporary: she expects a positive growth rate since technology has to recover its long run trend, and the interest rate increases. By contrary, in the other two cases, the attribution of a movement in γ_t pushes down the interest rate. Figure 2 also shows an important feature of the model: pessimism creates a wedge between the steady state of the interest rate and its long run average.

In Figure 3 we simulate the response to a permanent shock. Under complete information the interest rate drops on impact while for the Bayesian and for the ambiguity averse agent the response is much smoother, given that they assign a positive probability to the case of a transitory disturbance.

Finally, Figure 4 shows the negative effect on the interest rate of an increase in pessimism, as also clear from equation (39).

This exercise highlights two features of the core mechanism: first, the presence of ambiguity makes the response to a temporary shock to technology qualitatively similar to the response of a permanent shock in case of complete information. Second, pessimism (or ambiguity aversion) may act as an amplifier of a negative shock, increasing its persistence.

4.2 Identifying the sources of pessimism

In the simple endowment economy approximated to the first order it is not possible to distinguish what causes variations in pessimism: an increase in ambiguity aversion has the same effects on the risk-free rate of higher ambiguity. Note, however, that while α_t only affects the mean of the distorted posterior distribution, Q_t moves both the mean and the variance. Then, to identify the effect of these two different sources of pessimism we introduce a risky asset and consider an approximation to the second order: we can capture the extra precautionary motives related to an increase in ambiguity.

FIGURE TO BE ADDED

5 A particle filtering strategy for the approximated solution

We propose a particle filtering strategy for a model approximated with a second order series expansion. As in Amisano and Tristani (2007) we avoid using the bootstrap filter, which is not very efficient, and suggest a proposal density that is based on the linear approximation.

For simplicity consider the notation in which the vector x_t contains all the variables in the model, so that the solution is written as:

$$x_t \approx x_0 + qx_{1t} + \frac{q^2}{2}x_{2t} \tag{40}$$

where q is the perturbation parameters. The dynamics of the first order term are linear:

$$x_{1t} = \psi_q + \psi_x x_{1t-1} + \psi_\omega \epsilon_t \tag{41}$$

while the dynamics of the second order are non-linear:

$$x_{2t} = \psi_{qq} + \psi_x x_{2t-1} + \psi_{xx}(x_{1t-1} \otimes x_{1t-1}) + \psi_{xq} x_{1t-1} + \psi_{x\epsilon}(\epsilon_t \otimes x_{1t-1}) + \psi_{\epsilon\epsilon}(\epsilon_t \otimes \epsilon_t) \tag{42}$$

where $\epsilon_t \sim N(0, I)$. For simplicity let's assume that the parameters ψ's and the constant vector x_0 are known, and at each time t we observe the variables Y_t that are linearly related to x_t: $Y_t = Fx_t$.

Suppose that the posterior distribution of x_{1t-1} and x_{2t-1} is approximated by a set of particles $\left\{(x_{1t-1}, x_{2t-1})^{(i)}\right\}_{i=1}^N$ and associated weights $\left\{w_{t-1}^{(i)}\right\}_{i=1}^N$. At time t we observe the new data Y_t and we want to approximate the joint posterior distribution of x_{1t} and x_{2t}:

$$p\left(x_{1t}, x_{2t} | Y_{1:t}\right)$$

through a new set of particles.

Our state space model has a linear observation equation:

$$Y_t = Fx_0 + qFx_{1t} + \frac{q^2}{2}Fx_{2t} + v_t \tag{43}$$

where $v_t \sim N(0, \sigma_v^2)$ appears because equation (40) is an approximation. The state equation is non-linear, and it is described by equations (41) and (42). The peculiar problem of the system is that both x_{1t} and x_{2t} are approximations of the same vector x_t, so they depend on the same vector of shocks ϵ_t. Moreover ϵ_t enters non linearly in x_{2t}. In other words, even if equation (41) is linear, the model is not conditionally linear and we need to draw x_{1t} and x_{2t} jointly.

5.1 The proposal distribution

The proposal we have in mind comes from the following intuition: the last term of the approximation (40) is of higher order, it counts less. Moreover, if we had a first order approximation, that is if we used $x_t \approx x_0 + qx_{1t}$ instead of equation (40), the model would have been linear and Gaussian. Then, the idea is to use, as importance distribution, the posterior distribution we would have using a first order approximation. This posterior, which is available analytically, should not be too far from the posterior distribution of the second order approximation.

Consider the auxiliary model in which the observation equation is:

$$Y_t = Fx_0 + qFx_{1t} + T\frac{q^2}{2}Fx_{2t} + v_t \tag{44}$$

where T can be zero or one::

$$T = \begin{cases} 0 : \text{ auxiliar model} \\ 1 : \text{ original model} \end{cases}$$

As proposal distribution we use the following density:

$$p\left(x_{1t}, x_{2t}|x_{1t-1}, x_{2t-1}, T = 0, Y_t\right) = p\left(x_{2t}|x_{1t}, x_{1t-1}, x_{2t-1}, T = 0, Y_t\right) p\left(x_{1t}|x_{1t-1}, x_{2t-1}, T = 0, Y_t\right)$$

Note that given x_{1t} and x_{1t-1}, x_{2t} is deterministic, since the shock ϵ_t is determined by equation (41). Then, the first term on the right hand side of the equation above is equal to one. In the second term, instead, we can drop the dependence from x_{2t-1} since $T = 0$. Then:

$$p\left(x_{1t}, x_{2t}|x_{1t-1}, x_{2t-1}, T = 0, Y_t\right) = p\left(x_{1t}|x_{1t-1}, T = 0, Y_t\right) \tag{45}$$

which can be computed by the Kalman filter. Drawing from the proposal can be done as follows: first draw $x_{1,t}$ from the density in (45), this give the shock ϵ_t through equation (41), and we obtain x_{2t} using equation (42).

Using a resample-propagation scheme as in Pitt and Shephard (1999), the algorithm is:

1. Resample with weights proportional to: $\tilde{w}^{(i)} \propto w_{t-1}p\left(Y_t|g(x_{1t-1}^{(i)}, x_{2t-1}^{(i)}), T = 1\right)$ where $g(x_{1t-1}^{(i)}, x_{2t-1}^{(i)})$ is the expected value of $x_{1t-1}^{(i)}$ and $x_{2t-1}^{(i)}$

2. Propagate $(x_{1t}, x_{2t})^{(i)}$ from $p\left(x_{1t}|x_{1t-1}^{(i)}, T=0, Y_t\right)$

3. Compute new weights:

$$
\begin{aligned}
w_t^{(i)} &= \frac{w_{t-1}p\left(Y_t|x_{1t}^{(i)}, x_{2t}^{(i)}, T=1\right)p\left(x_{1t}^{(i)}, x_{2t}^{(i)}|x_{1t-1}^{(i)}, x_{2t-1}^{(i)}, T=1\right)}{w_{t-1}p\left(Y_t|g(x_{1t-1}^{(i)}, x_{2t-1}^{(i)}), T=1\right)p\left(x_{1t}^{(i)}|x_{1t-1}^{(i)}, T=0, Y_t\right)} \\
&= \frac{p\left(Y_t|x_{1t}^{(i)}, x_{2t}^{(i)}, T=1\right)p\left(Y_t|x_{1t-1}^{(i)}, T=0\right)}{p\left(Y_t|g(x_{1t-1}^{(i)}, x_{2t-1}^{(i)}), T=1\right)p\left(Y_t|x_{1t}^{(i)}, T=0\right)}
\end{aligned}
$$

6 Conclusions

For Oreste:

- Goal of the project: study the role of beliefs in accounting for the drop in interest rates after the Great recession: it is an empirical question.
- Not there yet: in the current draft we propose a strategy to answer our research question.
- The strategy has the following features:
  1. Ambiguity on the growth rate of technology
  2. Smooth ambiguity preferences to analyze the role of pessimism in a flexible way (possibility to distinguish two sources of pessimism)
  3. Second order approximation. Perturbation technique for general equilibrium model under smooth ambiguity preferences: using numerical methods would make the estimation not feasible
  4. Particle filter for series expansion.
- Next step: put the core mechanism in an appropriate model and estimate it: we are considering the characteristic of the model to be estimated.

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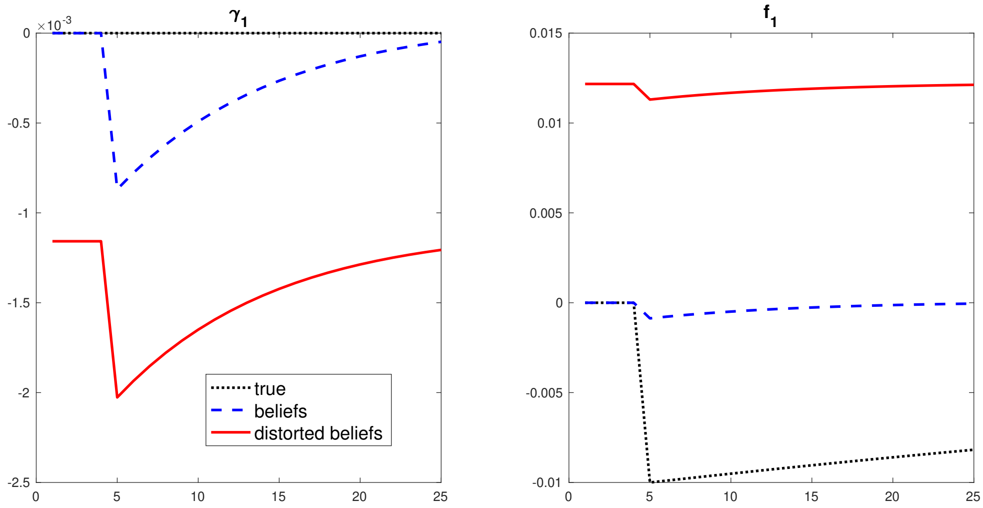

Figure 1: Dynamics of beliefs after a transitory shock to technology

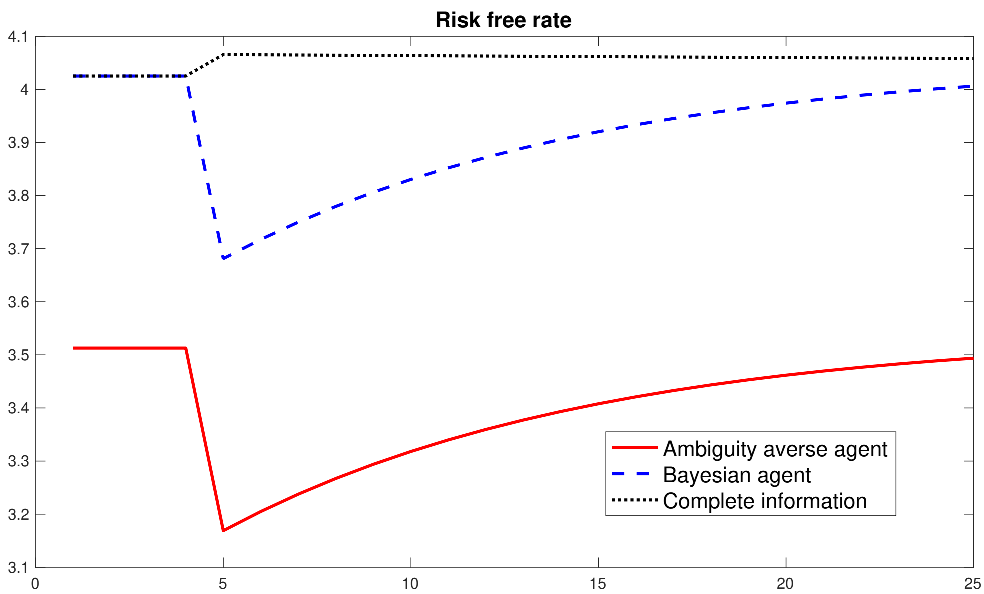

Figure 2: Impulse response to a transitory shock to technology

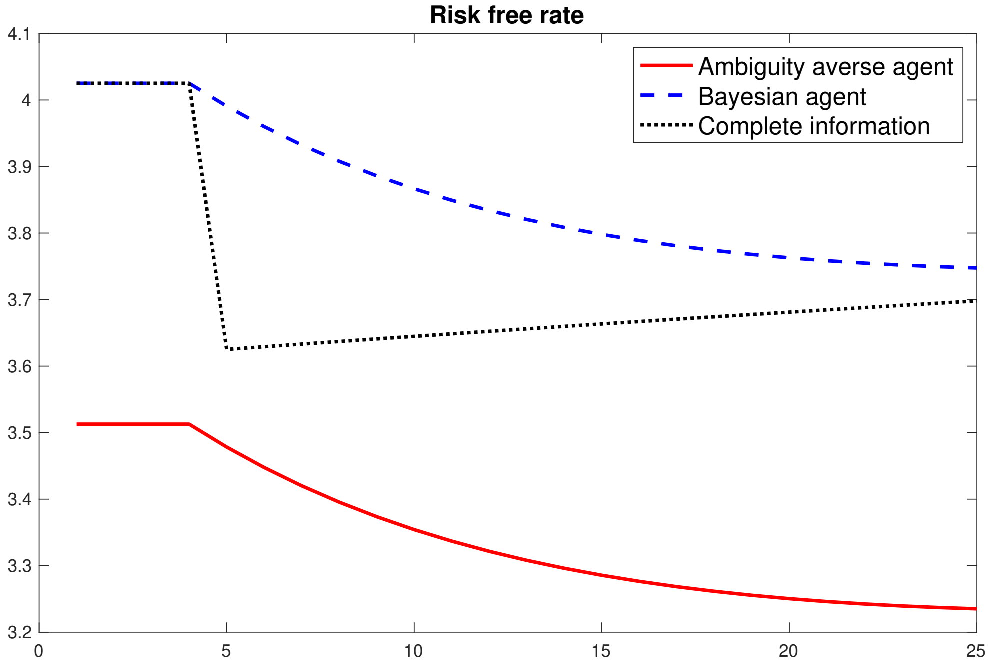

Figure 3: Impulse response to a permanent shock to technology

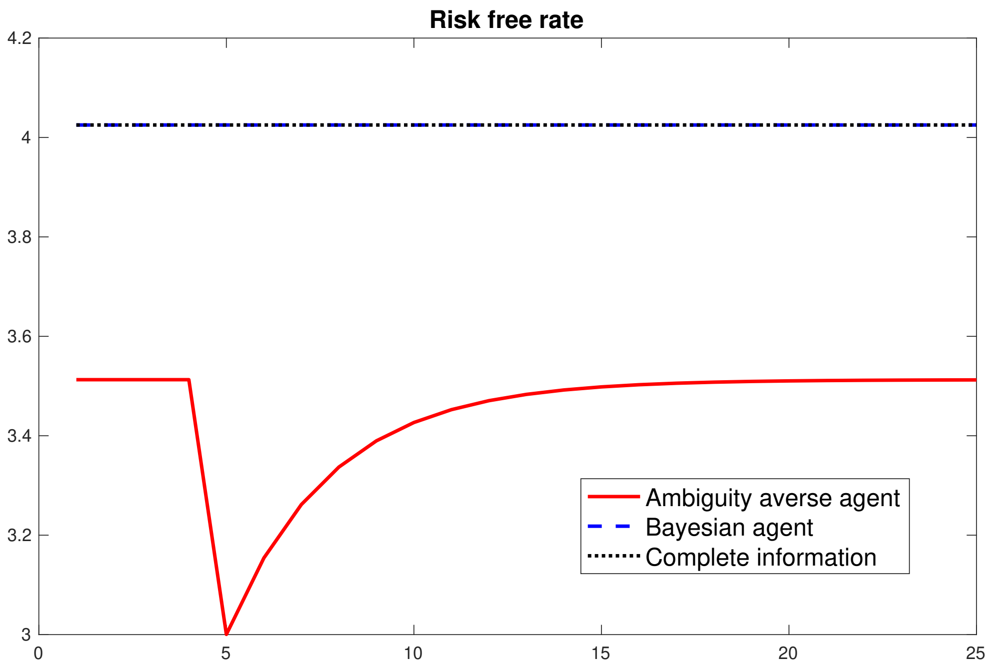

Figure 4: Impulse response to a shock to pessimism